
The Case for CAS

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Abstract

There are many arguments for and against the use of CAS. This book sets out to provide an argumentative case for the use of CAS. The first section outlines some of the key issues surrounding the use of CAS and the related research, which supports these findings. The next section details the ideas of teachers from various countries who already use CAS in teaching learning and assessment. It sets out their experiences in terms of lesson ideas, teaching sessions, and the adaptations which have to be made to question types and examination papers, to assess suitably. The final section of the book outlines the arguments for and against the use of CAS (“the Advocatus Diaboli”), providing what the authors believe is a convincing case for the use of CAS in the teaching and learning of Mathematics.

Foreword

The ‘CASE for CAS’ project began as an idea which emerged from T³ (Teachers Teaching with Technology) Europe, when the annual coordinators conference became less of an information exchanging meeting and more of a collaborative working forum. Since 2002, many of the T³ coordinators have been working on collaborative projects such as T³ on-line courses and TI Interactive. So why not a collaborative CAS Group? CAS presents some interesting challenges to the teaching, learning and assessment of mathematics, and indeed is banned in the national examinations in some countries such as Scotland and Germany. This project was born out of the notion that if teaching and learning with CAS is to be promoted, then it has to be embedded in the ‘high stakes’ assessment, - by ‘high stakes’ we mean assessment which is at the end of a course of study and counts for University entrance for example. How can a country be persuaded to release a ban on and adopt CAS? How can assessment be adapted to take into account the challenges of CAS? This book aims to provide persuasive arguments for the former question and provide full and illustrative examples of the latter.

The book summarises the research evidence to support the case for the use of CAS and takes the experience of teachers from different countries that already use CAS in teaching learning and assessment. It sets out their experiences in terms of lesson ideas, teaching sessions, and the adaptations which have to be made to question types in order to assess suitably. The book then takes time to counter the arguments in discussion (“the Devil’s Advocate”), finally providing what the authors believe are robust and watertight arguments which answer these counter arguments.

Since the idea first germinated contributions have been submitted from Austria, Australia, Belgium, Denmark, France, Scotland and Switzerland. The book thus represents the views of a number of teachers in widely differing situations with a wealth of experience of teaching with CAS - a truly international collaborative effort.

1. Introduction

Look around you in the tree of Mathematics today, and you will see some new kids playing around in the branches. They're exploring parts of the tree that have not seen this kind of action in centuries, and they didn't even climb the trunk to get there. You know how they got there? They cheated: they used a ladder. They climbed directly into the branches using a prosthetic extension of their brains known in the Ed Biz as technology. They got up there with graphing calculators. You can argue all you want about whether they deserve to be there, and about whether or not they might fall, but that won't change the fact that they are there, straddled alongside the best trunk-climbers in the tree – and most of them are glad to be there.

Kennedy, B. (1995)

Some Mathematics becomes more important because technology requires it.

Some Mathematics becomes less important because technology replaces it.

Some Mathematics becomes possible because technology allows it.

Waits, B. (2000)

The two initial quotations are provocative in nature and were chosen to reflect the flavour of the work to follow in this book. As Michael Meagher stated in his discussion paper *The arguments for the use of CAS as an effective tool in supporting teaching and learning of mathematics are well established. Now comes the real work.* Meagher, M. (2001). The “real work” is demonstrating the connectivity between curriculum and assessment and most crucially demonstrating assessment using CAS. This book aims to test the ‘robustness’ of the arguments over what is taught using CAS and how this is assessed. The background and theory sets out the research results already available in a convenient and succinct form to allow ready access to data indicating four key issues:

- That the teaching of mathematics becomes more interesting with CAS;
- That students are more interested and motivated to learn mathematics with CAS;
- That students who use CAS are at least as good in ‘pencil and paper’ skills (PAP) as their traditional counterparts;
- That high stakes assessment and CAS are compatible.

2. Background and research

Michael Meagher may well feel that the arguments for use of CAS as an effective tool are well established, but at this point in the book it is useful to reflect on just how well established these arguments are. Given the titles of numbers of papers considering the use of CAS such as *Education or CAStration*,³ *Symbolic Manipulation on a TI-92 – New threats or hidden treasures*,⁴ and *Indispensable Manual Calculation Skills in a CAS-Environment*,⁵ it is clear that not everyone in the mathematical world agrees. Indeed an article in the recent issue of *Mathematics Teacher* stated *More than with any other technology, thinking about incorporating computer algebra systems (CAS's) in secondary mathematics raises major concerns* (NCTM, 2002).

The research evidence outlined below, considers the use of CAS and is broken down into sections addressing four key issues, enabling the main thrust of available research to be set out, prior to looking at particular teaching sections and consequent assessment.

THE TEACHING OF MATHEMATICS BECOMES MORE INTERESTING WITH CAS STUDENTS ARE MORE MOTIVATED TO LEARN MATHEMATICS WITH CAS

In 1998 the Advanced Calculators and Mathematics Education (ACME) Review stated that, *the Group is firmly of the view that effective use of graphics and CAS calculators can have a number of important benefits. These benefits include improving students' perceptions of mathematics and consequently increasing their enjoyment of the subject and motivation to learn* (SCCC, 1998). This was a considered statement made after considerable research from various countries as well as directly by word of mouth from students themselves.

After the publication of the ACME review, research carried out in Scotland indicated that *A positive impact was noted, acknowledging a motivational effect as well as an enhancement in pencil and paper (PAP) activities and ... many students appreciating the use of the calculator in their own time, assisting with homework, familiarising themselves with the interface and in so doing, raising their confidence with mathematics* (Macintyre and Forbes, 2002). Students themselves made the following comments:

Oh no, better having it all the time. When I used it to check was at home when I didn't have a teacher.

It helped me through the higher; gave me more confidence and did more for me than the '83.

A number of studies support this point of view. One study in the USA remarked, *At the conclusion of last year I gave my students a survey regarding the use of CAS in the classroom to see if their perception of doing mathematics had changed. 65% of the students indicated that using CAS gave them more confidence and enhanced their ability to learn math, while no students said it gave them less confidence* (Longhart, 2002). In Germany (Thuringia) a recent survey aimed to find out the views the students have on the use of the TI-89 in mathematics courses. The study, conducted over a period of 3 years, sought the opinions of grade 11 and 12 students on such questions as *working with the TI-89 is no problem for me*, and *The TI-89 makes me feel more confident when solving problems*.

The students in the survey particularly liked being able to work faster and to check results. The students, who indicated a moderate enthusiasm for mathematics generally, showed an overall positive attitude to the CAS. (Schmidt and Moldenhauer, 2002)

Extensive work carried out by the Austrian ACDCa, www.acdca.ac.at, substantiated this change in attitude and motivation. In two very large projects, (70 classes in 44 secondary schools with some 2000+ students), carried out between 1993 and 1998, research was carried out into the teachers' and students' acceptance of the new teaching methods using CAS technology. They focused, among other things, on changes in student's success in learning, on changes in their attitude towards mathematics and science, and on differences in knowledge, comparing traditional teaching methods as opposed to CAS supported methods.

Students commented on a positive approach to mathematics, perhaps more generated by the group approach to problem solving, which teaching with CAS necessitates, rather than the use of CAS itself. Nevertheless students commented very enthusiastically on change in attitudes to maths *although I am not a good mathematician, the many various ways to tackle a problem were very interesting, great and motivating for me* and *Now we are sure that we have a very sound knowledge about that stuff because we have learnt that by ourselves*. The global impression of the project was viewed by students as excellent and most wished to carry on with this method in the future.

In conclusion of this section it is notable that we have concentrated on what the students thought of learning with CAS. We know, however, that motivated teachers means motivated students and the change in teaching style that teaching with CAS necessitates. *New Work Plans in the Classroom* (Kendal, Stacey and Pierce, 2001) provides an interesting insight into what changes are needed and on the move towards more student centred learning required. The international literature suggests that these changes in teaching style and classroom environment are likely to impact favourably on students' achievement and motivation. The leader of the ACDCa project Helmut Heugl summarised by saying:

We did not find such a "Holy Land" when working with CAS, not for the pupils and not for the teachers, but an interesting learning environment for a

- *more meaningful,*
- *more interesting,*
- *and more future oriented mathematics education.*

Giving the 'last word' in this section to a student: *Overall I found it a hard year, but it was very informative. If I had to choose between an 'ordinary' maths teaching and this one, undoubtedly I would take this one* (Heugl, 2002).

STUDENTS WHO USE CAS ARE AT LEAST AS GOOD IN 'PENCIL AND PAPER' SKILLS AS THEIR TRADITIONAL COUNTERPARTS

There is a widely held perception by many teachers of mathematics that *CAS use will hamper development of, or lead to atrophy of, by-hand symbolic manipulation skills* (NCTM, 2002). Indeed a highly polarised view in the UK deems *narrow pragmatism, and the associated reluctance to analyse and to discuss principles, continues to undermine attempts to understand the lessons of the past and to improve future mathematics teaching in England. In particular, there is a real danger that we may repeat with CAS the errors we made in taking for granted the use of ordinary calculators* (Gardiner, 2001).

There are two inherent dangers in ignoring the development of CAS. Firstly, many students will feel that the subject is out of touch with the real world where such tools are used widely. They may thus be put off studying mathematics more widely. Secondly, students who use them are likely to develop an unthinking dependence on CAS to do the simplest algebraic operations. If we accept that there is a case for use of CAS in the learning of mathematics then there is a strong case for taking full advantage of what CAS can offer and accepting the challenge that they present to the school mathematics curriculum and how it is presented. *The core of the controversy is less about whether students should use them (CAS calculators), but rather when they should use them...* (French, 1998).

Most teachers agree that fluency with symbolic manipulation is vital for progress in mathematics. Indeed it is this very fluency which provides access to general cases and exact solutions. What is less in agreement is the degree of fluency and thus emphasis given to this within the curriculum. *A person fluent with a particular algebraic concept or principle has three distinctive and defining characteristics:*

- *They can decide when it might be useful to use this concept or principle;*
- *They have the symbolic skills to do it correctly;*
- *They know what the significance of the end product is.*

We suggest that too much time has been spent traditionally on the second of these and too little time on the first and last (Kissane, Bradley and Kemp, 1997).

Learning the lessons of the past, considered classroom use of any calculator involves an emphasis on the development of mental and written methods working alongside critical use of the calculator. *It is important to strike the right balance between the development of the necessary computational and algebraic skills, and the use of technology to free students from routine drudgery. The aim should be to let calculators shoulder the burden of subsidiary calculations, hence enabling students to tackle more realistic and significant problems. Teaching should also become more effective because there should be more time to give greater emphasis to the selection of appropriate strategies and interpretation of results* (SCCC, 1998). This balance of PAP and use of CAS will be explored in more detail in the main part of the paper.

What is the evidence that use of CAS will lead to the atrophying of basic skills? In short, many years of research have shown this to be untrue. *A number of research studies have investigated students' performance on achievement tests that measured computational or procedural skills after mathematics instruction that incorporated CAS. All but one of the fifteen studies that we reviewed found that students whose courses included a CAS did just as well on test items that required computational and procedural skills as those students whose instruction did not include this technology. Of course, we might expect the CAS-using students to do well if they used CAS on the comparison tests. When a CAS was not used on these tests, eight of nine studies reported that although by-hand computations and procedures were not emphasised in the related classroom instruction, students who used CAS's during instruction did as well or better on the procedurally oriented achievement test as those who did not use the technology* (Heid, Blume, Hollebrands and Piez, 2002). This is compelling evidence indeed.

Other studies outside the USA have come to similar conclusions.

In 1996/7 in Norway a study was carried out with students using CAS. This was a small study with two groups of students studying mathematics using CAS. For the pupils from the test groups: 41 examination papers - the average mark was 3.90. For the ordinary pupils: 155 examination papers - the average mark was 3.48. These tests included items where only pencil and paper was allowed. The

results replicated the findings of the US studies. *Even though the number of the participants in this experiment is too low to draw any certain conclusions, does the survey above give us relatively strong signals of what to expect if we start using the symbolic calculator in the mathematics training. There seemed to be documentation for following claims:*

- *The calculation skill increases.*
- *Motivation for working with the subject increases.*
- *The ability of problem solving is stimulated.*
- *The professional understanding increases, and then it becomes easier to find solutions in professional problems.*

(Aarstadt 1997) www.rogaland-f.kommune.no/~strand/symb_calc.htm

In Scotland as a direct result of the (ACME, 1998) recommendations, a pilot study was carried out between 1990 and 2001 to gather ‘home grown’ information about the use of CAS. Very little if any use of CAS had been made in Scottish schools prior to this study. Three pairs of schools were chosen to have dedicated access to CAS and three schools chosen as ‘control’ schools, the latter having access to graphing calculators but not CAS. At the beginning of the study all students were given an ‘algebraic skills’ PAP test. The students were given a similar PAP ‘skills’ test at the end of the study, with the second test having some common items to the initial test. The students had no access to CAS in the final examination and this had a profound effect on use as the year progressed. 356 students sat the initial skills test and 227 of these could be traced on the follow up test. *Having considered the empirical data gathered through the two algebraic skills assessments there is evidence to suggest a difference in performance across the schools. The intervention schools performed significantly better at the end of the study on the matched questions, those repeated from earlier in the year as well as on the new questions that were pitched towards the year 12 topics* (Macintyre and Forbes, 2001).

A study similar to the Scottish pilot and conducted in Montana in 2000, concluded that *Contrary to popular belief, we found that using CAS to complete basic symbolic manipulations does not erode the retention of basic skills. To the contrary, the Math 3 students that employed CAS retained or improved their symbolic manipulation skills.*

On a slightly different note and by comparison, was the study carried out at the US Military Academy (USMA) during the 1999/2000 academic year. This study sought to investigate the impact of the integration of the TI-89 CAS on students’ achievement and attitudes. Unlike most other studies which compared users to non-users, this study looked at changes in technology users after integrating a new tool (the TI-89 as opposed to the HP 48G or others without “qwerty”) with additional capabilities. Again, unlike other studies, the students in this one at the USMA were permitted to use the TI-89 in the final exam. Once again the results supported the assertion that appropriate use of CAS can enhance the teaching and learning of mathematics and lead to improved *procedural, conceptual and application performance* (Connors and Snook, 2001).

The fact that using CAS does ‘no harm’ to students’ ‘basic symbolic manipulation skills’, however, does not in itself provide a complete case for using CAS. Many studies have looked at the effect of using CAS on students’ mathematical skills and their understanding of concepts. *A range of studies showed that the overall conceptual understandings of the students who used a CAS were the same as or better than the understandings of their counterparts who did not use a CAS. Other studies showed that students who used a CAS better understood particular concepts – variable and function – than*

those who did not use a CAS. From these studies, we conclude that CAS use enables students to develop deeper conceptual understanding” (Heid, Blume, Hollebrands and Piez, 2002).

Giving the second last word in this section to the ACDCA project, Heugl summarises nicely the advantages of using CAS. *Mathematical competence opens doors to productive futures. A lack of mathematical competence closes these doors. But the vision of mathematics teaching and learning is not the reality in the majority of classrooms and schools. In this age of information technology students need to learn a new set of mathematics basics that enables them to solve problems creatively and resourcefully. The National Council of Teachers of Mathematics in USA (NCTM) formulated guidelines for educational decisions laid out in principles and standards. One of these principles is called the „Technology Principle“: Calculators and computers are reshaping the mathematical landscape, and school mathematics should reflect these changes. Students can learn more mathematics more deeply with appropriate and responsible use of technology. The investigations of the Austrian CAS projects justify an optimistic point of view. The major results are not especially mathematical contents - it is a more pupil centred, experimental way of learning. In other words, the new tool supports all of the four key qualifications.... CAS-supported mathematical education supports and encourages the 4 key qualifications:*

- *Subject competence*
- *Methodological competence*
- *Social competence*
- *Personal competence*

much better than traditional mathematics education (Heugl, 2000).

The very title of the article *“New emphasis of fundamental algebraic competence and its influence in exam situation”* opens the discussion about standards and expectations of students and their expected behaviour, particularly in high stakes exam situation. *“Indispensable Manual Calculations in a CAS Environment”* - what are they? (Herget, et al, 2000).

The last word in this section goes to Australia.

What are the Future Goals for Mathematical Proficiency in the CAS-Age?

If computer algebra systems (CAS) are to be permitted in upper secondary mathematics teaching, careful consideration should be given to the balance between by-hand and by-CAS procedures (Flynn, Berenson and Stacey, 2002).

What should students record when solving problems with CAS?

There are people who believe that the calculator acts like a black box, producing answers without any understanding being required by the students and there are others who believe that a very good understanding of mathematics is required in order to use the calculator (CAS) to solve problems. From our experiences....we would argue that the latter is definitely the case (Ball and Stacey, 2001).

www.edfac.unimelb.edu.au/DSME/CAS-CAT

The answer to these two questions is fundamental to the whole issue and leads us to the final section of this research summary.

HIGH STAKES ASSESSMENT AND CAS ARE COMPATIBLE

The answers to these questions may well depend on the country concerned and whether or not the examinations are centralised or not. Whatever the nature of the examination structure as Drijvers says, *as soon as assessment gets involved, however, attitudes seem to be less favourable. Teachers are*

afraid that students will not develop “hand skills” when technology enters the examination room. They may fear that students will pass the examination just by button pressing, and that testing becomes unfair because of different access to new technologies. In many countries a debate is going on about the role of new technologies at examinations. School authorities, examination boards and teachers are struggling with the question of how to deal with these powerful instruments. Different policies are followed and so far there is no convergence (Drijvers, 1998).

The situation has changed quite markedly in some countries with regard to CAS in assessments, and progress has been made in some countries, since Drijvers paper was written. Austria now has a well-developed approach to assessment, albeit different in form from prior to CAS. Countries like Denmark have developed central examinations consisting of two parts, Pencil and Paper (PAP) and an ‘anything goes’ paper. Slovenia has looked at alternative examinations for the Matura to see if this can be adapted to accommodate the use of CAS by students. (Lokar and Lokar, 2001) In England between 1994 and 1996, working groups were set up to assess the potential impact of CAS on the A-level curriculum and examinations. (Monaghan, 1997) More recently the CAS-CAT project in Melbourne Australia has produced a number of influential papers and solutions to what has otherwise seemed like an impasse.

Yet still countries like Scotland and Germany retain the ban on CAS (at least one state (Thuringia) is piloting the potential of CAS – a start!). The only major study conducted in Scotland found that *concerns over the ‘Black Box’ aspect of technology have been unfounded. Indeed, given appropriate conditions and encouragement, the use of CAS as an investigative tool is something that was well received by the schools in the study. The full potential of such technology however, will not be made apparent until schools are encouraged to utilise the resource through a change in policy on assessment. Regardless of personal visions and expertise in harnessing the power of the technology, staff are going to be discouraged and students themselves will be reluctant to make use of CAS calculators whilst they remain ‘banned’ in assessments* (Macintyre and Forbes, 2002).

The ACME Group (SCCC, 1998) recognised the barrier of assessment during its deliberations and an early recommendation was to adopt non-calculator papers (implemented in Scotland at ages 16 + from 2000) in order to ‘free up’ the use of technology. Little progress has been made since then. The Group also recognised that the “assessment tail tends to wag the curriculum dog!” Perhaps it is now time to take a radical look at what we want the students of today to learn in mathematics and THEN find a suitable way of assessing them.

The ACDCA Projects recognised this early on in their deliberations and it is perhaps from them we can learn. ***Which is the more valid question?***

Do the new ways of mathematics learning and teaching influence the exam situation?

or

Does the exam situation influence new ways of learning and teaching? (Heugl, 2000)

There is little doubt that in the past the exam situation has always had a great influence on the content and the teaching of mathematics education. Previously, the emphasis was on teacher led lessons and process based assessment. Times in mathematics have changed! Assessment must change also!

A fundamental principle formulated by the American NCTM is called the ***Assessment Principle***. Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.

- Assessment should be more than merely a test at the end of instruction to gauge learning.
- Teachers should be continually gathering information about their students.
- Assessment should focus on understanding as well as procedural skills.
- Assessment should be done in multiple ways, and teachers should look for a convergence of evidence from different sources.
- Teachers must ensure that all students are given the opportunity to demonstrate their mathematical learning.

The ACDCA CAS projects recognised that the traditional way of assessment was not suitable to the new ways of learning which they observed in their CAS classes.

Some significant changes of the learning process in our CAS-classes which strengthen the necessity of changes in the exam situation:

- *A more pupil-oriented learning process. More frequently mathematical discussions among the students. The teacher is not the source of knowledge; he supports the independent acquisition of knowledge by the students.*
- *Experimenting, the trial- and error method: We seldom find the “algorithmic obedience” where the teacher shows one way which all the students then accept and follow*
- *Working in pairs or groups can be seen much more frequently.*
- *Beside the teacher there exists now a new, very competent expert – the tool CAS. That means pupils do not always need the teacher for examining the correctness of their ideas and results.*
- *Phases of “open learning” where the students are individually organising the velocity and the contents of their learning process.*
- *The CAS is not only a calculation tool, students can also store knowledge by defining modules or using the text editor. Therefore it is senseless to forbid the use of learning media, like books or exercise books during the tests.*
- *New emphasis on fundamental competence. A shift from calculation skills to other competence and skills.*
- *A clearer emphasis on problem solving*
- *A more application oriented mathematics.*
- *More frequent cross curriculum teaching.” (Heugl, 2000)*

The main section of this paper now looks at these ideas in detail to examine how a number of different countries have developed these ideas in teaching sequences and the consequent changes in assessment which result.

3. Countries involved and current ‘CAS’ status

3.1 CAS status in Austria

The situation in Austria is characterized by a very early start in the CAS-era. With little tradition in the use of graphic calculators in Austria, the use of CAS in math education and in secondary schools started with the purchase of a national license for Derive in the early nineties. This was initiated by the Austrian government, the late Eduard Sziruscek from the Ministry of Education and Helmut Heugl, who continues to be a major force in maths education in Austria.

In 1992 the first DERIVE-conference in Krems brought together math educators from all over the world for the first exchange of experience related to the use of CAS in secondary schools and at university entrance level. Helmut Heugl and a group of enthusiast teachers founded the ACDCA (Austrian Centre of Didactics of Computer Algebra) to encourage teachers to modify their teaching practice to incorporate use of the new tools. Since then four nationwide ACDCA projects have been launched with the aim of supporting teachers in this new CAS era. In the initial years activities mainly focused on training teachers how to use CAS and on the production of attractive teaching materials. Previous and current ACDCA projects include investigations into the impact of CAS-supported teaching on the culture of problems and teaching, the effect of CAS on assessment and the shift in competencies for students and teachers (e.g. competence of methods, competence of recognizing structures and patterns, competence of designing and grading open question problems, ...).

All of these activities were supported by the Austrian government, the Pedagogical Institutes and school authorities in the federal states.

When the TI-92 entered the market the projects were opened for handheld CAS and the number of involved students and educators increased immediately, because CAS could be used without having PCs available. Cooperation with T³ (“Teachers Teaching with Technology”) proved to be very fruitful and made many pre- and in service courses possible.

At the moment we estimate that 25-30% of secondary school students (age 15 – 19) have access to CAS and this percentage is continuously growing. Additionally textbook authors and publishers have started including CAS in their textbooks. The fact that Austria does not have common externally set examinations and that teachers have wide scope in curriculum delivery allows for experimentation in the use of CAS in the classroom and for assessment.

The results of the four ACDCA-projects and numerous CAS-oriented papers can be downloaded from the website www.acdca.ac.at.

3.2 CAS status in Belgium

Secondary education in Belgium is divided into three levels where each level is studied over a two year period. The math curriculum in Belgium has supported the use of graphic calculators in the second level (age 15-16) since September 1999. Nowadays (2003), working with a graphic calculator is widely accepted by teachers and there are textbooks available which incorporate the use of graphic calculators. There are no central math examinations for secondary schools in Belgium.

In 1998, Belgian T³-EUROPE instructors (Teachers Teaching with Technology) commenced CAS-training for teachers in the use of TI-92's. This provided impetus for some didactical projects and experiments related to teaching secondary school mathematics with handheld CAS. However, the extent of CAS use is very dependent on the school and the math teacher. There are some teachers working with CAS (handheld or DERIVE) in a math-oriented education where students have 6 or 8 hours of math weekly.

From September 2004 there will be a new math curriculum for the third level (age 17-18) and it is likely that the use of CAS will be encouraged in that curriculum. Currently students in the second level are allowed to use graphic calculators and textbooks assume that students will have a graphic calculator. It may be more difficult to convince teachers and parents that students in the third level should learn mathematics with access to CAS.

3.3 CAS status in Denmark

General upper secondary education in Denmark ("Gymnasium") leads to the final examinations, Baccalaureate ("Studentereksamen"). University entrance is based upon results from these examinations. The written examinations are national and the Ministry of Education monitors the examination system. The written examinations in mathematics consist of two parts, paper and pencil ("PAP") short response questions and extended response questions. For the latter students can use books, notes, tables of formulae, graphing calculators, etc. Since the mid 1990's there has been an alternative written examination, where CAS is allowed for part 2. The majority of questions are identical in the two papers. If a teacher applies to the Ministry of Education and the pupils agree unanimously then permission to take the alternative CAS examination will be given. In the CAS examinations pupils are allowed to use CAS calculators such as the TI-89/92Plus/Voyage 200, or a computer with a CAS programme such as MathCAD/TI InterActive!/Derive.

Although written work is the main focus of this book, it is worth mentioning that the Danish Studentereksamen also has oral examinations for mathematics. Orals are considered an important part of the assessment process in the "Gymnasium", making a broad assessment possible.

In this book we show the 2000 Mathematics Studentereksamen in Denmark, both the PAP paper, and the CAS assumed paper and the solutions are provided for the CAS paper.

In the near future it is expected that CAS will be assumed for all written mathematics examinations in the "Gymnasium", however there will still be a PAP component of the examinations.

3.4 CAS status in Scotland

The current situation in Scotland can be summarised in one short statement. Graphing calculators are allowed in examinations but CAS machines are not.

There was a major review conducted in 1998, Advanced Calculators and Mathematics Education (SCCC, 1998) www.svtc.org.uk/sccc/acme which sought the views of interested parties and made recommendations for the future. The review reflected the fact that teachers overwhelmingly endorsed the use of graphing calculators. Opinions were however divided on the use of CAS. The major issue appears to centre over the use of CAS in the central examinations. This despite the fact that these examinations now have two papers, one of which relies on pencil and paper only.

The examination authority, Scottish Qualifications Authority (SQA) is now actively promoting graphing calculator solutions to certain questions in the second paper. It recently (November 2002) issued advice to schools outlining expected solutions to selected questions. There is no apparent movement to relax the ban on CAS. This despite 'home grown' evidence to suggest that use of CAS does not have a detrimental effect on student's 'basic skills' (MacIntyre and Forbes, 2002). It seems

that the central examination is the barrier to exploiting the full potential of CAS. *Regardless of personal visions and expertise in harnessing the technology, staff is going to be discouraged and students themselves will be reluctant to make use of CAS calculators whilst they remain 'banned in assessments* (MacIntyre and Forbes, 2002). This paper aims to provide the necessary weight of evidence to convince those in power to remove the ban.

3.5 CAS status in Switzerland

The Swiss school system has some peculiarities. First of all, a population of about 7 million is split into four official languages and secondly, the country has an extremely decentralised system, resulting in 26 different autonomous education departments (cantons) with only minimal influence from the central government. The process of learning how to use technology in mathematics teaching has a rather lengthy history going back to the 1980s.

Essentially there are three phases:

- a) In the 1980's there were trials by pioneer teachers developing programs for algorithms to carry out numerical approximation on desktop computers.
- b) In 1992 a large conference took place in Baden (3 days). Demonstrations of Mathematica, Maple and Derive at workstations and graphing calculators sparked lively discussions, not only among the participants but also in the wider mathematical education community. Following the conference, several schools ordered site licences for Mathematica or Maple and permitted the use of CAS-systems in the "ghetto of computer rooms". Derive and graphing calculators were only used by a few schools. One notable exception was the vocational secondary schools, which integrated graphing calculators into the mathematics curriculum.
- c) In 1996 the first secondary school classes were equipped with CAS handheld calculators (TI-92). At the same time the program of T³-Switzerland became active. Many Swiss secondary school teachers recognised that CAS handheld technology provides the opportunity for a new challenging way of teaching. The situation after 1996 resembles the spread of an infection. In spite of the immunity of the universities against this infection, the CAS virus spread throughout the country. According to an investigation in 2002 more than 60% of secondary II students in Switzerland are using CAS or graphing calculators making Switzerland the leading European country in this domain. It is clear, however, that an efficient integration lags behind this impressive number in many cases. In order to reach the less innovative teachers there is an urgent need for new textbooks and problem sets adapted to good practice in combining the potential of handheld CAS with the need to shape young brains. Activities in this direction are on the way. We hope that this book will be another milestone in this direction.

4. Classification scheme for examination questions

We propose a scheme for classifying our questions with respect to the role a CAS plays in the solution. Necessarily in practise any attempt to classify is faced with the possible fuzziness of the categories used.

We distinguish five categories:

- C0 Exercises where the use of CAS is of little or no help. More importantly, the typing would take more time than solving the problem by-hand (this of course depends on the students' skill).
- C1 Traditional exercises (by-hand or using scientific calculators) that are solved faster or even trivialised by CAS (emphasis usually on manual skills).
- C2 Exercises that essentially test the ability of using CAS competently.
- C3 Exercises starting from traditional ones that are extended to CAS-exercises (e.g. by including formal parameters or using realistic data).
- C4 Exercises that are difficult or time consuming to solve without CAS or those that can only be solved with the aid of CAS.

Reference: Lokar M. & M.: *CAS and the Slovene External Examination. In "The International Journal of Computer Algebra in Mathematics Education", 2001. Vol. 8, No 1.*

Note: Most of the exercises in this book are of type C3 and C4. Such exercises often lead to new methods of teaching (investigations, group work etc.).

The following simple examples illustrate the classification:

Example of C0:

Most limits of a single expression, e.g.: $\lim_{x \rightarrow \infty} \frac{8x^7 - 5x^4 + 9}{4x^7 + 3x^6 - 2x}$.

(Of course, to be faster by hand you need some training with such limits)

Example of C1:

Find the intersection points of the two graphs of the functions $f(x) = 4 - x^2$ and $g(x) = (x-1)^2$.

Solution: The intention of this exercise is to practice the formula for solving quadratic equations.

The CAS trivialises the answer through `solve(f(x)=g(x), x)`.

Example of C2:

Determine the zeros of the function

$$f(x) = x^{10} - 6x^9 - 11x^8 + 24x^7 + 34x^6 - 36x^5 - 46x^4 + 24x^3 + 29x^2 - 6x - 7.$$

- a) in exact mode
- b) in approximate mode

Interpret the results.

Solution: a) zeros ($f(x), x$), result: $\{-1, 1, 7\}$ (exact result).

b) Result: $\{-1.019, -0.955, -0.988, 0.998, 0.999, 7.000\}$.

$f(x) = (x-7)(x-1)^4(x+1)^5$, hence the graph touches the x -axis flatly. It seems that the algorithm stops as soon as $|f(x)|$ is smaller than a given value and goes on to the next interval.

Example of C3:

Introduce parameters into the example for C1:

a) $g(x) = (x-a)^2$

For which values of a do the graphs of $f(x)$ and $g(x)$ have just one intersection point?

b) Change furthermore $f(x)$ to $f(x) = b - x^2$.

What is the locus of all osculating points of the curves $f(x)$ and $g(x)$?

Solution: a) $a = 2\sqrt{2}$ b) $y = x^2$

Example C1 is trivialised by CAS. By introducing parameters one can transform the example to a non-trivial CAS-example (Bruno Buchberger: Trivialisation is not trivial, *Visiteme* 2002).

Example of C4: Cebotarev Assumption

(Group or distributed individual work)

Factor $x^n - 1$ for $n = 1, 2, \dots, 10$ with integer coefficients (on voyage 200: `factor(x^n-1)`). All coefficients of the polynomial factors are 1. Is this true for all exponents n ? Each student or group has to verify for some values of n between 11 and 200.

Solution: The problem was set by the Russian mathematician N.G. Cebotarev in 1938 in the Russian journal "Progress in mathematical sciences (volume IV)" and solved by V. Ivanov in 1941. The first exponent which leads to integer coefficients > 1 is $n = 105$. The voyage 200 factors the expression in 8 seconds, but the problem cannot be solved by-hand by high school students.

(From an old book: I.S. Sominski: *Die Methode der vollständigen Induktion*, DVB, Berlin 1974. With a hint of my colleague Reto Schuppli.)

5. Examples to illustrate the change in teaching with CAS Teaching sequences and consequent assessment

5.1 Ages 15-17 – Parabolas

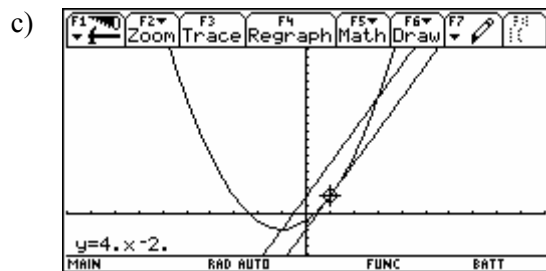
Parabolas 1

- (1) Given $f(x) = x^2 + 2x - 1$:
- Calculate the slope of the secant line through the points $P(-1, -2)$ and $Q(3, 14)$ on the graph of f .
 - Calculate $f'(1)$.
 - Illustrate the result graphically.
 - Now do the same question with $P(1, 2)$ and $Q(4, 23)$.
 - Calculate $f'(2\frac{1}{2})$.
 - Prove that $f'(x) = \frac{f(x+h) - f(x-h)}{2h}$ for any x and h .
- (2) Prove the result in (f) for any polynomial $f(x) = ax^2 + bx + c$.

Solution: (partial)

(1) a) $\frac{14 - (-2)}{3 - (-1)} = 4$.

b) $f'(1) = 2 \cdot 1 + 2 = 4$.

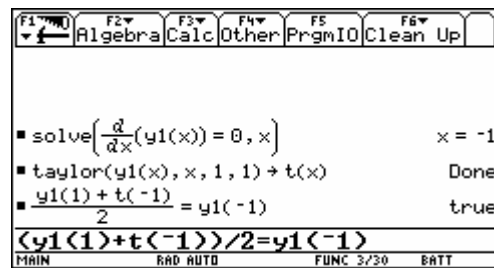
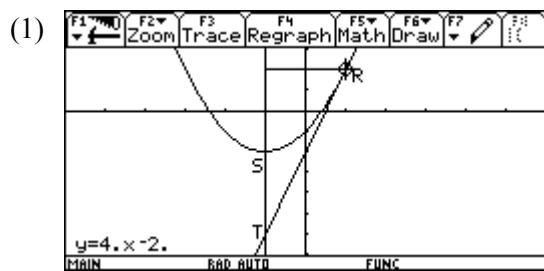


(2)

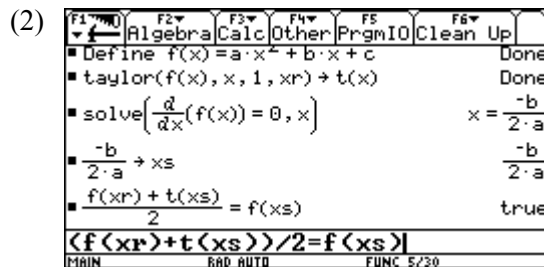
Parabolas 2

- (1) Given $f(x) = x^2 + 2x - 1$:
- Determine the coordinates of the summit S of the parabola.
 - Determine an equation of the tangent line t at the point $R(1, 2)$.
 - Determine the coordinates of the point T of intersection between the tangent line t and the axis of symmetry of the parabola.
 - Compare the y -coordinates of the points R , S , and T . Conjecture?
- (2) For any point R on a parabola $y = ax^2 + bx + c$ prove your conjecture about the vertical position of the points R , S (the summit) and T (the point of intersection between the tangent line at R and the axis of symmetry of the parabola).

Solution: (partial)



Note: First screen: The equation of the tangent line is found by using a graphic tool (F5).
 Second screen: The equation of the tangent line is found by using a CAS tool (Taylor polynomial of degree 1).



Note: By hand, the algebra is easier if you translate the parabola to the origin. With CAS there is no problem.

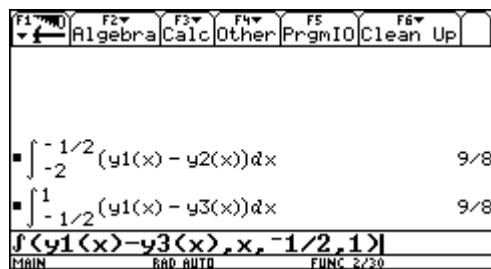
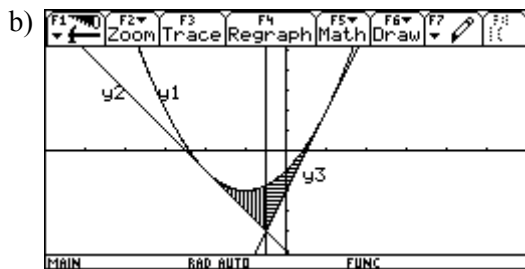
Parabolas 3

- (1) Given: $f(x) = x^2 + 2x - 1$:
- Find the abscissa x_p of the point of intersection P between the tangent lines at $x_1 = -2$ and $x_2 = 1$. Conjecture?
 - Calculate the area of the region determined by the parabola, the tangent at $x_1 = -2$ and the vertical line $x = x_p$.
 Calculate the area of the region determined by the parabola, the tangent at $x_2 = 1$ and the vertical line $x = x_p$.
 Conjecture?

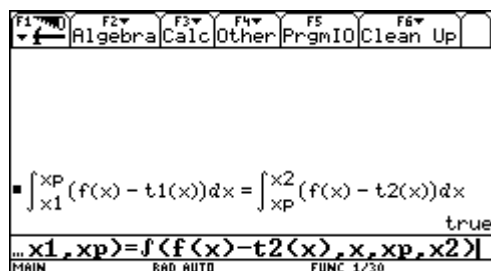
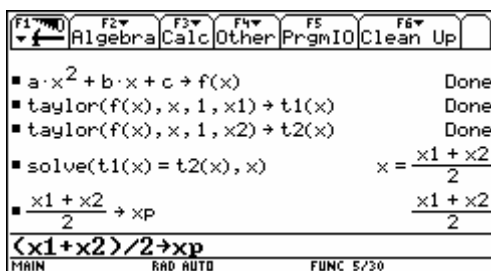
- (2) Given: $f(x) = ax^2 + bx + c$:
- Find the abscissa x_p of the point P of intersection between the tangent lines at any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the parabola.
 - Prove that the vertical line $x = x_p$ divides the region between the parabola and the tangents into two regions with equal areas.
 - Prove that for the parabola $y = x^2$ the coordinates of P are $\left(\frac{x_1 + x_2}{2}, x_1 \cdot x_2\right)$.

Solution: (partial)

(1) a) $x_p = -\frac{1}{2}$.

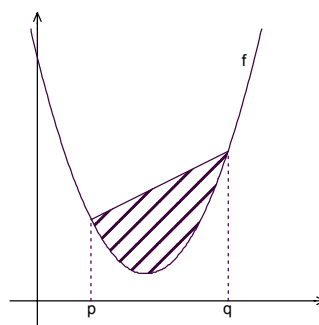


(2)



Exercise

Prove that for any parabola $y = f(x) = ax^2 + bx + c$ the shaded area does not depend on the values of p and q , but on $p-q$ (and a) only.



5.2 Ages 15-17 – Greatest common divisor and least common multiple A plea for pure mathematics

Using the [F2]-Menu you can factor a number into its prime factors. Let's do this for the large number 844074000. Set your device in Exact or Auto Mode!



factor(844074000) =

- (1) Using this tool you can easily find the GCD of 844074000, 4765246200 and 45585540000. Repeat first how to find the GCD of two or more small numbers.

Hint: You can apply factor on a list of numbers.

844074000 =

4765246200 =

45585540000 =, hence GCD = =

At this moment students are using a well known algorithm for finding GCD however access to CAS enables them to deal with larger numbers and expressions than usual.

If a number is the GCD of two or more numbers, then the quotients of the given numbers and the GCD shouldn't have any prime factors in common. Check this for the previous example.

Store the GCD in your CAS: [STO] gcd.

Do you receive an error message? Try once more but use another name because gcd is obviously a reserved function name. So take: [STO] cd.

844074000 : cd = = (factor !)

4765246200 : cd = =

45585540000 : cd = =

Hint: A clever use of a list makes the process more comfortable!)

True? (yes / no)

If your answer is **no** then try to find the mistake!

- (2) Can we prove the statement given in the box above?

This is a chance for the teacher to present a very easy to follow indirect proof. In earlier lessons we showed, in a short introduction into logic, that $a \rightarrow b$ is equivalent to $\neg b \rightarrow \neg a$.

We want to show: If $\text{GCD}(a,b) = T$ then $\text{GCD}\left(\frac{a}{T}, \frac{b}{T}\right) = 1$.

We show that $\text{GCD}\left(\frac{a}{T}, \frac{b}{T}\right) \neq 1$ implies that T is not the GCD of a and b .

Suppose $\text{GCD}\left(\frac{a}{T}, \frac{b}{T}\right) = p \neq 1$, then $\frac{a}{T} = p * \alpha$ and $\frac{b}{T} = p * \beta$.

So $a = T * p * \alpha$ and $b = T * p * \beta$ which implies that $T * p$ must be a "greater" common divisor than T . And this is obviously a contradiction to the fact that $\text{GCD}(a,b) = T$.

Find and check in a similar way the GCDs of:

(3) $\text{GCD}(328563a^3 b^4 c^5 d^2, 247104a^2 b^5 c^4, 93085200a^4 b^2 d^6) = \dots\dots\dots$

(4) $\text{GCD}(900(a+b)^3 (a-b)^2 (a^2 - b^2)^2, 1000(a^2 + b^2)(a^2 - b^2)^2 (a^3 - b^3), 500(a^3 - b^3)^2 (a^3 + b^3)^2 (a^2 - b^2)^2) = \dots\dots\dots$

Comment: It seems that it is not possible to factor general expressions and the coefficients in one operation. Try to find an alternative!

(5) $\text{GCD}(36a^6 + 12a^5 b - 47a^4 b^2 - 14a^3 b^3 + 12a^2 b^4 + 2ab^5 - b^6, 54a^8 - 27a^7 b - 63a^6 b^2 + 34a^5 b^3 + 8a^4 b^4 - 7a^3 b^5 + a^2 b^6, 18a^5 b^2 - 39a^4 b^3 + 20a^3 b^4 + 6a^2 b^5 - 6ab^6 + b^7)$

GCD = $\dots\dots\dots$

These examples will force the students to check their data. They have to edit the bulky expressions very carefully and copy the results onto paper.

The device “knows” the function `gcd(number1, number2)` for calculating the GCD of *number1* and *number2*. Check this using two randomly chosen pairs of numbers.

(6) Calculate $\text{GCD}(844074000, 45585540000, 4765246200)$ and compare with (1)!

It was interesting to observe students trying to find a way out of that trap. Eventually they discovered the concept of a nested function and immediately tried to extend this for four or more numbers.

How could you use `gcd` for three or more numbers? $\dots\dots\dots$

(7) What is the meaning of the abbreviation `gcd`?

$\dots\dots\dots$

Have you tried `gcd` for non-integer numbers? Do that now and report your results:

$\text{GCD}(\dots\dots\dots) = \dots\dots\dots$

$\text{GCD}(\dots\dots\dots) = \dots\dots\dots$

$\text{GCD}(\dots\dots\dots) = \dots\dots\dots$

Here we encourage the students to generalize a well known rule in an unusual way. They are asked to interpret the outcomes.

Explain the outcome. Can you find a rule?

I assume that students used only rational numbers. Now consider at least one irrational number:

GCD(.....) =

- (8) I also assume that students have worked with decimal numbers. It is "more mathematical" to work with fractions:

$$\text{GCD}\left(\frac{3}{4}, \frac{1}{2}\right) = \dots\dots\dots$$

$$\text{GCD}\left(\frac{3}{4}, \frac{1}{2}, \frac{3}{5}\right) = \dots\dots\dots$$

$$\text{GCD}\left(\frac{3}{4}, \frac{1}{2}, \frac{3}{5}, \frac{17}{3}\right) = \dots\dots\dots$$

The conjecture is that the result is a fraction $\frac{1}{\text{lcm}}$ (all denominators).

Does your conjecture also hold for $\text{GCD}\left(\frac{3}{4}, \frac{6}{5}\right)$?

Now we are learning that our first idea might not hold. We have to refine the interpretation..

Is a new interpretation necessary?

Can you find a problem which needs $\text{GCD}\left(\frac{3}{4}, \frac{6}{5}\right)$ for its solution?

The rule is $\text{gcd}(\text{numerators})/\text{lcm}(\text{denominators})$. It is the greatest fraction which gives integer results for the divisor for the given numbers.

Problem: Given is a part of a wall 0.75×1.20 which should be covered by tiles in form of a square of maximum area. How many tiles of which size are necessary?

Solution: $\text{GCD}\left(\frac{3}{4}, \frac{6}{5}\right) = \frac{3}{20}; \left(\frac{\frac{3}{4} \cdot \frac{6}{5}}{\left(\frac{3}{20}\right)^2} \right) = 40.$

We need 40 tiles with side length $3/20 = 0.15$.

We can assume that the students know how to find the LCM of two (smaller) numbers.

- (9) The next task is to find the least common multiple for the numbers from (1) by-hand: (using factor () is allowed)

LCM (844074000, 4765246200, 45585540000) =

The main task in the next paragraph is to find a check for LCM and to explain the process and check the validity of any assumptions made.

- (10) Consider a check similar to that for the GCD. (Refer to task (2)).

Describe the process:

Confirm your conjecture using the numbers from (9). Try to find the proof by yourself.

Find the LCMs for the expressions from (3), (4) and (5). Check the result for at least one of them:

(11) LCM(problem 3) =

(12) LCM(problem 4) =

(13) LCM(problem 5) =

Task (14) is not so demanding for English speaking students. But in Austria we call the LCM "kleinstes gemeinsames Vielfaches" with the abbreviation kgV. So this task forces them to use the manual and look carefully for the required CAS function.

(14) Which function of your calculator returns the LCM of two numbers? Try it and apply again the nested function concept. Check one of the results of (9), (11), (12) or (13)

.....

(Use the manual!)

(15) Try to solve the following problem.

Wine is bottled in bottles to 0.5 litres. It shall be poured into other bottles of 0.35 litres content. How many bottles à 0.35 litres must be used that no rest will remain in the 0.5 l bottles?
(provided by René Hugelshofer)

(16) What is the meaning of LCM for rational numbers?

(17) Try again (15).

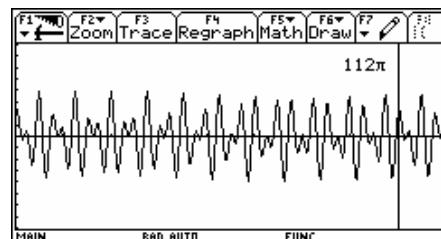
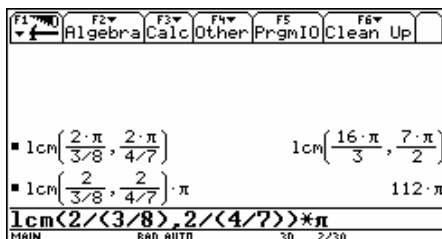
Solution: $\text{LCM}\left(\frac{1}{2}, \frac{35}{100}\right) = \frac{7}{2}$; $\frac{7}{\frac{1}{2}} = 10$ and $\frac{7}{\frac{35}{100}} = 20$

10 bottles of 0.35 litres are filled by 7 bottles of 0.5 litres.

The LCM of two fractions has another fine application which unfortunately cannot be explained at this level. (Idea from Guido Herweyers).

What is the period of $\sin\left(\frac{3x}{8}\right) + \cos\left(\frac{4x}{7}\right)$?

Solution:



- (18) The LCM of two expressions a, b can easily be derived from the $\text{GCD}(a, b)$ and the two expressions. Find this relationship!

Work systematically! Start your investigation using smaller numbers and try to find a rule.

$$a = \dots\dots\dots; b = \dots\dots\dots, \text{GCD}(a, b) = \dots\dots\dots, \text{LCM}(a, b) = \dots\dots\dots$$

$$a = \dots\dots\dots; b = \dots\dots\dots, \text{GCD}(a, b) = \dots\dots\dots, \text{LCM}(a, b) = \dots\dots\dots$$

$$a = \dots\dots\dots; b = \dots\dots\dots, \text{GCD}(a, b) = \dots\dots\dots, \text{LCM}(a, b) = \dots\dots\dots$$

$$a = \dots\dots\dots; b = \dots\dots\dots, \text{GCD}(a, b) = \dots\dots\dots, \text{LCM}(a, b) = \dots\dots\dots$$

There are so many patterns in mathematics and from my point of view recognizing patterns can very often help students solve their maths problems, so we need to give students many more opportunities to practice this ability to recognise patterns.

The next step is to express the pattern in words, then as a rule or formula.

When you are sure you know the relationship you should check your conjecture using two "larger" numbers or expressions from the 1st sheet:

You should be able to express the relationship as a formula:

$\text{GCD}(a, b) =$

Express this formula in words:

Until now the conjecture has been an unproved conjecture and not a theorem. The proof is still lacking. Try to carefully follow the exact proof shown (*Eugenio Roanes gave a sketch of this proof after a talk in Gettysburg.*).

Take two numbers $a = 8400$ and $b = 1440$

$$a = 2^4 \cdot 3 \cdot 5^2 \cdot 7 \text{ and } b = 2^5 \cdot 3^2 \cdot 5 \cdot (7^0)$$

$$\text{GCD}(8400, 1440) = 2^4 \cdot 3^1 \cdot 5^1 \cdot 7^0 \text{ and } \text{LCM}(8400, 1440) = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7^1$$

$$\text{GCD} * \text{LCM} = 2^{4+5} \cdot 3^{1+2} \cdot 5^{1+2} \cdot 7^{0+1} = a * b$$

Now we will generalize the process shown above.

$$a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots\dots\dots \cdot p_n^{\alpha_n} \text{ and } b = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \dots\dots\dots \cdot p_n^{\beta_n}$$

with $p_n =$ greatest prime factor occurring. If p_k does not occur in one factorization then α_k or $\beta_k = 0$).

$$\text{GCD}(a, b) = p_1^{\min(\alpha_1, \beta_1)} \cdot p_2^{\min(\alpha_2, \beta_2)} \cdot \dots\dots\dots \cdot p_n^{\min(\alpha_n, \beta_n)}$$

$$\text{LCM}(a, b) = p_1^{\max(\alpha_1, \beta_1)} \cdot p_2^{\max(\alpha_2, \beta_2)} \cdot \dots\dots\dots \cdot p_n^{\max(\alpha_n, \beta_n)} \text{ hence}$$

$$\begin{aligned} \text{GCD}(a, b) * \text{LCM}(a, b) &= p_1^{\min(\alpha_1, \beta_1) + \max(\alpha_1, \beta_1)} \cdot p_2^{\min(\alpha_2, \beta_2) + \max(\alpha_2, \beta_2)} \cdot \dots\dots\dots \cdot p_n^{\min(\alpha_n, \beta_n) + \max(\alpha_n, \beta_n)} \\ &= p_1^{\alpha_1 + \beta_1} \cdot p_2^{\alpha_2 + \beta_2} \cdot \dots\dots\dots \cdot p_n^{\alpha_n + \beta_n} \end{aligned}$$

This was a welcome opportunity to present an exact proof using mathematical notation. It was helpful to refer to the example above at each stage of the proof so that students could follow the process.

Then I had the- crazy - idea to find another generalization: is it possible to extend the theorem for more than two numbers?

The students and I encountered a real challenge in exploring and investigating the next problem. We were very surprised to find a solution using trial and error.

- (19) The above theorem might imply the consequence that we could generalize for three numbers/expressions: $a * b * c = \text{LCM}(a,b,c) * \text{GCD}(a,b,c) ??$

Check if this statement is true or not for two examples of your choice.

- (20) Try the numbers 12, 30 and 64.

Are there any triples of numbers for which the formula holds???

- (21) Try for (12, 18, 25).

- (22) See if the formula holds for (150,210,330).

Explain your results.

Sometimes you can have a "mathematical feeling" but it is not so easy to express that feeling in words. It is very important to practice the ability to explain mathematics.

- (23) Here is a real challenge.

Try to find a relation between GCD and LCM which holds in all cases:

The students found one solution:

$$a b c = \frac{\text{LCM}(a,b,c) \cdot \text{GCD}(a,b) \cdot \text{GCD}(a,c) \cdot \text{GCD}(b,c)}{\text{GCD}(a,b,c)}$$

At the time when I was busy with this paper in school I was running a T^3 seminar for Secondary School teachers and I presented this paper as an example for working with the TI-92. At 10 30 in the evening my phone rang and a colleague proudly told me that she had found a solution:

$$a b c = \text{LCM}(a,b,c) \cdot \text{GCD}(a,b,c) \cdot \text{LCM}(\text{GCD}(a,b), \text{GCD}(a,c), \text{GCD}(b,c)) .$$

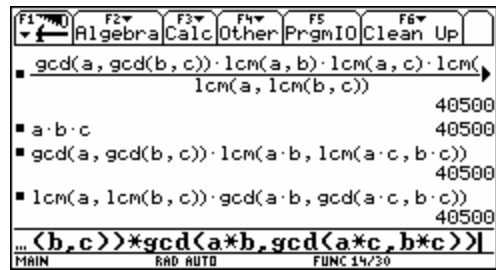
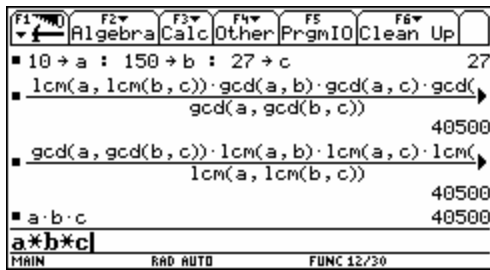
I wrote an e-mail to my friend Hannes Wiesenbauer and asked him for an alternative solution. He replied: "That problem is not so easy to solve, but there is a solution:" and he added a theorem which leads to

$$a b c = \text{GCD}(a,b,c) \cdot \text{LCM}(ab, ac, bc)$$

$$a b c = \text{LCM}(a,b,c) \cdot \text{GCD}(ab, ac, bc)$$

The two formulae show a wonderful symmetry and we could observe the principle of duality which the students had met a couple of weeks before when they had a short introduction into set theory and logic.

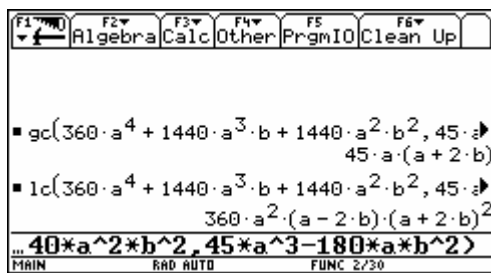
You might imagine that it did not last very long and they asked if there was a dual formula for their product and fortunately there was.



(24) Unfortunately there is no function in the TI-92 which allows students to find the GCD and LCM of general expressions. So now we will try to develop such a function.

Consider two functions gc and lc , which are working in a way presented on the TI-screen shown below. You should find the GCD and LCM of the expressions

$$360a^4 + 1440a^3b + 1440a^2b^2 \quad \text{and} \quad 45a^3 - 180ab^2$$



As gc and lc are related very closely (see (18)):

.....

It will be sufficient to find a function for one of them.

Hint: Do you know any application of GCD and/or LCM?

We will stick to the LCM.

The TI-function `getdenom(fraction)` gives back the denominator of a fraction.

Try this for the following fractions:

What are the denominators of the fractions: $\frac{2}{5}$, $\frac{2a}{5b}$, $\frac{a^2+b}{b^2+a}$, $5a$, $\frac{3}{4} \cdot \frac{2}{9}$

Use [CATALOG] to find the respective function which returns the numerator of a fraction:

.....

Check it:

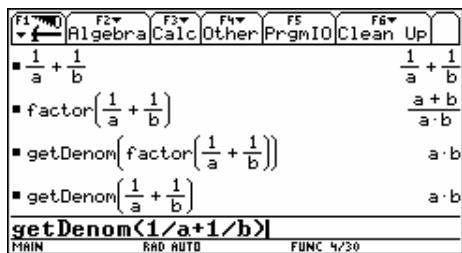
`getdenom` will support our task to find the function lc !

Are you ready?

Do you have a formula, which returns the LCM of two expressions t_1 and t_2 ?

Then define your function in the following way:

define $lc(t_1, t_2) = \dots\dots\dots$



You can follow the learning process to extract the denominator of a sum of fractions, which seems to form the lcm of the two expressions t_1 and t_2 . The students learned to develop and finally to define a self made function (and later to use it). It is very interesting to observe the strategies of the students. See a possible way to achieve the solution.

Check the validity of your function by applying it to the two expressions from the screenshot.

Does your function work?

Congratulations! But to make sure, perform just one more little test.

Check your function for two numbers: $lc(3, 6) = \dots\dots\dots$

Is everything ok? Why not?? What has happened??

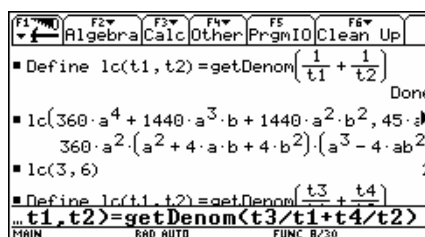
Find an explanation and improve the function.

Students should learn not be satisfied with their results too early. It makes sense to check the validity of rules even for very trivial cases.

Improved version (Introducing auxiliary variables t_3 and t_4 to avoid cancellation):

define $lc(t_1, t_2) =$

.....



We can use one function to define a new one, which gives:

define $gc(t_1, t_2) = \dots\dots\dots$

Closing comments

I like applied mathematics but I am sure that we can not do without pure mathematics. I found that the students liked investigating the problems presented and I spent a lot of time during the breaks answering their questions (What if, Why not? Am I right? Do you think?....). I didn't hear the question: What is all that stuff good for? I was teacher at a vocational school!!! So I think we can tempt pupils to have their "Adventures in Their Heads". It is important to explain what they have done, for example: "Now you found an important rule on your own!", "This is your first self made function!",... The students are really proud of their mathematical achievements and their selfconfidence is growing. There were some very emotional moments when I felt that the sparkles were spreading and when I then left the classroom I felt something like: "That is it, I've got them". These are the precious moments when we learn why we have become teachers. I know that the use of a CAS in whatever form is not a necessary condition, but it is helpful in providing a new motivation for students and, for teachers too. We should not let these unique chances for learning pass us by. What I have shown here using the TI-92 is platform independent. I am sure that many chapters of our traditional curricula can be changed and that it is not necessary to add many new topics as a

consequence of the use of modern technologies. On the contrary, I must say, that it is likely that we now need more time in the classroom than before, because mathematics teaching has a much wider scope than ever before. New technologies not only open new fields for school mathematics and enable us to tackle interesting applications but may also sometimes help us to find a way "back to the roots".

Two Assessment Examples

The students used their self defined function kgv to check the common denominator for calculations with fractions and later for solving equations. Students also did this using traditional methods simple problems.

- (1) This example shows how to combine indispensable basic skills and computer algebra.
(C4 at this level, because factoring the denominators is too time consuming.)

$$\frac{6}{x^3 + 4x^2 - 9x - 36} - \frac{2}{2x^3 + 19x^2 + 59x + 60} = \frac{9}{2x^3 + 5x^2 - 18x - 45}$$

a) Use the TI to factor the denominators.
b) Find the domain.
c) Solve the equation by-hand.

The second example demonstrates that using a computer algebra system addresses much more than manipulation skills.

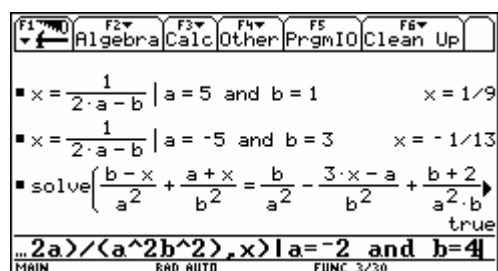
- (2) Symbolic Algebra in Equations (C3)

$$\frac{b-x}{a^2} + \frac{a+x}{b^2} = \frac{b}{a^2} - \frac{3x-a}{b^2} + \frac{b+2a}{a^2b^2}$$

a) Enter the equation and explain the TI's "simplification".
b) Give the equation's general solution and state the necessary conditions for the solution to be valid.
c) Find two special solutions!
d) Write down the output from the TI.
e) Explain the second part of the output and give an example to show why this output was obtained.

Answers (for question 2):

- a) Do the first part by-hand (bring over a common denominator)
b) $x = \frac{1}{2a-b}$ with $a, b \neq 0$ and $b \neq 2a$
c) $(a = 5, b = 1) \rightarrow x = 1/9$
 $(a = -5, b = 3) \rightarrow x = -1/13$



- d) Copy the expression
- e) For $b = -2a$ the equation turns out to be an identity.
- f) This needs manual calculation:
 Multiplication of the equation by a^2b^2 and
 later $x(4a^2 - b^2) = 2a + b | : (2a + b) \neq 0$

The calculator screen shows the following steps:

$$\frac{(a^2 - b^2) \cdot x + a^3 + b^3}{a^2 \cdot b^2} = \frac{-(3 \cdot a^2 \cdot x - a^3 - 2 \cdot a^2 \cdot b)}{a^2 \cdot b^2}$$

The user enters the equation into the solver:

$$\text{solve}\left(\frac{(a^2 - b^2) \cdot x + a^3 + b^3}{a^2 \cdot b^2} = \frac{-(3 \cdot a^2 \cdot x - a^3 - 2 \cdot a^2 \cdot b)}{a^2 \cdot b^2}, x\right)$$

The result is:

$$x = \frac{1}{2 \cdot a - b} \text{ or } \frac{2 \cdot a + b}{a^2 \cdot b^2} = 0$$

At the bottom of the screen, it says: solve(ans(1), x) LIVERP RAD AUTO FUNC 20/30

5.3 Ages 15-17 – Heron’s formula

The following tested teaching sequence will show you:

- how students are able to prove Heron’s formula, a rather hard task for most of the students without CAS-calculator,
- how Heron’s formula can be used as a functional module (functional thinking),
- how the usual pure calculation exercises about Heron’s formula can be replaced by nice and demanding ones, which are leading to reflection and a deeper understanding of mathematics,
- how a new dynamic is given to tasks by parametrisation.

5.3.1 Worksheet: Heron’s Formula

A triangle can be constructed with three sides given; therefore its area is also determined. The Greek mathematician Heron of Alexandria found the following formula for calculating the area of a triangle from its sides:

Heron’s formula for the area of a triangle

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{with } s = \frac{a+b+c}{2} \quad (\text{half of the circumference})$$

Exercise 1 (Category C1)

Store the above formula as a function named heron (a, b, c).

Attach the additional condition with |.

Calculate the area of the right triangle with sides 3, 4, 5.

Calculate the area of some other triangles (choose the length of the sides yourself).

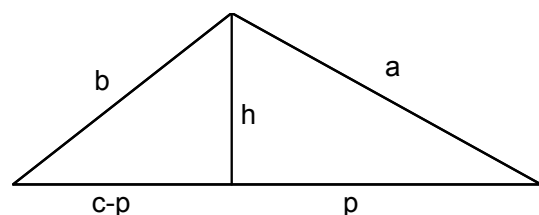
Interpret possible error messages.

Calculate the side c of a triangle from $\text{area} = 5$, $a = 3$ and $b = 4$.

Storing the formula as a function and applying it like a blackbox is a very efficient method when using CAS.

Exercise 2

Prove Heron’s formula using the figure beside and the following guidelines
(insert your results in the worksheet)!



Proof of Heron's formula

An elementary algebraic proof of Heron's formula is rather sophisticated and is difficult for students to do without help. The following proof was done individually with CAS and the following worksheet.

Follow the instructions.

a) We calculate the square of Heron's formula $area^2 = \text{heron}(a, b, c)^2$ in order to avoid roots.

The CAS-calculator returns (s has been replaced by $\frac{a+b+c}{2}$):

$$area^2 = \frac{-(a+b+c) \cdot (a+b-c) \cdot (a-b-c) \cdot (a-b+c)}{16}$$

which has to be proven.

b) For the area of the triangle, we start with a well-known formula using baseline c and height h :

$$area = \frac{c \cdot h}{2} \quad \text{or} \quad area^2 = \frac{c^2 \cdot h^2}{4}$$

c) c is given. Express h^2 in terms of a and p and then in terms of b and $c-p$ using the Pythagorean theorem:

$$h^2 = \quad \text{(using } a \text{ and } p, \text{ store this in } \text{hsqr})$$

$$h^2 = \quad \text{(using } b \text{ and } c-p)$$

d) Equate both expressions for h^2 and solve with respect to p .

$$\text{Check your answer: } p = \frac{a^2 - b^2 + c^2}{2c}$$

e) Factor hsqr from c) and substitute for p from d) (CAS: `factor() | p=.`).

Store the result in hsqr ($= h^2$) and insert it in $area^2 = \frac{c^2 \cdot \text{hsqr}}{4}$ from b).

Compare your answer with the expression obtained in a).

5.3.2 Further Exercises

Exercise 3 (Category C3)

Calculate the area of a trapezoid with sides $a = 6$, $b = 4$, $c = 3$ and $d = 3$.

Define a general function `trapez(a, b, c, d)`.

Exercise 4 (Category C4)

Does there exist a triangle with sides x , $x + 1$, $x + 2$ and area $T = 10$?

For which given area T does a solution exist? (An example from my colleague H.R. Schneider)

Exercise 5 (Category C4)

Does a triangle exist with sides 3 , x , $2x$ and area $T = 10$?

Does a triangle with the same sides exist for $T = 1, 3, 0$?

For which given area T does a solution exist?

Store the function `heron(3, x, 2x)` `STO>` `y1(x)`, draw the function and determine an approximate domain and range.

Draw the vertices C of possible triangles ABC with $|AB| = 3$. On which curve do these vertices lie?

You can also construct the locus using a dynamic geometry system.

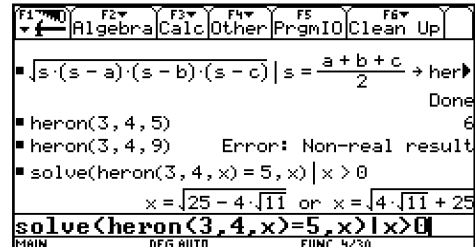
Exercise 6 (Category C0)

Who was Heron of Alexandria and what were his most important achievements? Where is Alexandria? What was Alexandria renowned for? Search for information to answer these questions in a dictionary, on the Internet or using other sources.

5.3.3 Solutions to the exercises

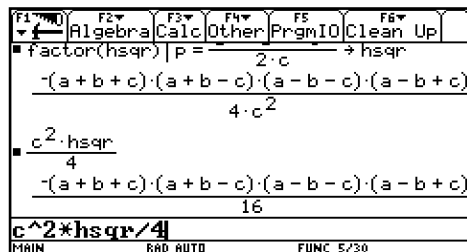
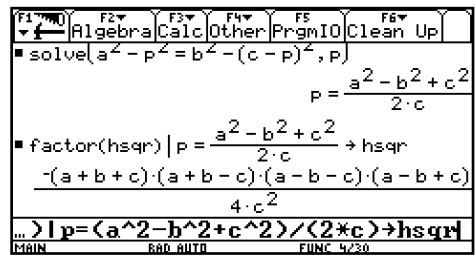
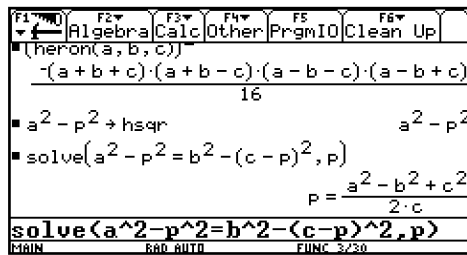
Exercise 1

If one side is larger than the sum of the others, a real solution doesn't exist. The latter example shows you that there exists at most 4 solutions to the equation, where two of them correspond to a triangle ($x > 0$). It is worth drawing the triangles, determining the height by calculation.



Exercise 2

You can see the steps of the proof on the screenshots below.



The proof with CAS is much easier than by hand, because the most difficult part is to transform the last expression to the form of Heron's formula. CAS simplifies Heron's formula automatically to the last expression. By the way, CAS shows you, how to perform the proof in an easier way by-hand as well.

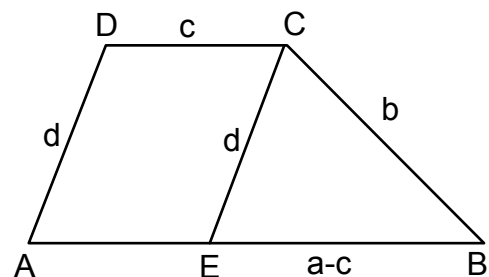
Exercise 3

We calculate the height of the trapezoid using the triangle EBC .

$$\text{solve}(\text{heron}(\text{abs}(a-c), b, d))$$

$$= h \cdot \text{abs}(a-c) / 2, h)$$

Then we calculate the area by $\frac{(a+c) \cdot h}{2}$.

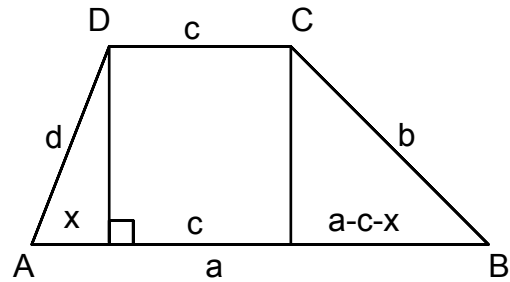


This general formula is very extensive and can be stored as a function $\text{trapez}(a, b, c, d)$, e.g. $\text{trapez}(3, 3, 6, 4) = 6\sqrt{5}$. Why does the function fail for rectangles!

A student proposed another solution without using absolute values (similar to the proof of Heron's formula): $d^2 - x^2 = b^2 - (a - c - x)^2 (= h^2)$.

Solve this equation for x and substitute x into the formula for the trapezoid:

$$\frac{(a+c) \cdot h}{2} = \frac{(a+c) \cdot \sqrt{d^2 - x^2}}{2}$$



Exercise 4

`solve(heron(x, x+1, x+2)=10, x)` has the solution $x = 4.018$ (and a negative one).

The CAS is overtaxed with `solve(heron(x, x+1, x+2)=t, x)` (equation of order 4).

With geometrical considerations you can see that for large values of x a triangle always exists and is nearly equilateral. At the other end you get a triangle with area 0 when $x = 1$. For $x > 1$ a triangle always exists with continuously increasing area.

You can also show the numerical effects of rounding, e.g. with $c = 0.00001$, the result is false.

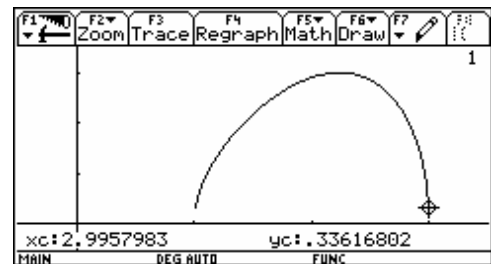
Exercise 5

`solve(heron(3, x, 2x)=10, x)`

Result: false

For $T = 1, 3, 0$ you get 1.109 or 2.962, $\sqrt{5}$, 1 or 3 (positive values only).

Using trace or the table function you can determine the domain $[1,3]$ and the range $[0,3]$.



The vertices C of possible triangles with $A = (0;0)$ and $B = (3;0)$ are lying on the Apollonian circle over the segment \overline{AB} with ratio 1:2. This is the circle of Thales over the segment with endpoints $(2;0)$ and $(6;0)$.

Note: This exercise can be done at an elementary level (9th school year). I did it after having introduced the Apollonian circle as an application of the intercept theorems. It can also be used as an experimental introduction to the Apollonian circle.

Exercise 6

Heron of Alexandria was a Greek scientist. We are not sure about his exact date of birth but we do know that it is between 150 B.C. and 250 A.C. Most important works: *Mechanika*, *Pneumatika*. Alexandria lies near the delta of the Nile in Egypt and is considered the most important scientific centre of the Hellenistic age. The Alexandria library was very famous with nearly one million book rolls.

5.3.4 Exam questions

a) Category C1

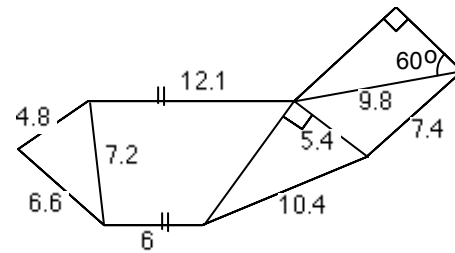
Use Heron's formula to find the area T of the triangle with vertices $A = (13; 2)$, $B = (5; 17)$ and $C = (22; -4)$.

b) Category C1

Use Heron's formula to calculate c and the area T of a triangle with $a = 7$, $b = 5$ and $h_c = 3$.

c) Category C1

In a consolidation of property a farmer can exchange his agricultural land (see figure beside, measures in km) for a square piece of land.



What is the length of the sides of this land?

d) Category C1

Does a triangle exist with sides 1, 2, x and area 10?

Draw `heron(1, 2, x)` `STO>` `y1(x)`.

Use 'Trace' to find an approximate domain of $y1(x)$ ($x > 0$). State the reason why x has to lie in this domain (geometrical consideration).

What is the largest possible area? Give a reason for your answer!

e) Category C1

Let us consider triangles with sides 4, x , and kx ($k \geq 1$).

For $k = 3$, determine the locus of the vertices of these triangles (baseline 4) and the largest possible area. (Tip: First consider triangles with area 0.)

What is the locus of the vertices for an arbitrary value for k ?

What is the locus for $k = 1$?

5.3.5 Solutions to the exam questions

a) Store the vertices of the triangle: `[-13, 2]` `STO>` `a`, `[5, 17]` `STO>` `b`, `[22, -4]` `STO>` `c`.
`heron(norm(b-a), norm(c-b), norm(c-a))`, result: 316.5.

b) First calculate side c : `zeros(heron(7, 5, c) - 3*c/2, c)`, result (numeric) `{2.325, 10.325}`. We then get T with: `3*c/2|c={2.325, 10.325}`, result `{3.487, 15.487}`.

Note: The command `zeros` returns a list, which allows you to calculate several values for a variable simultaneously.

c) If the modules `heron(a, b, c)` and `trapez(a, b, c, d)` are not still stored in the CAS, get students to define them again.

First calculate the missing side, s , of the trapezoid: `10.4^2 - 5.4^2` `STO>` `s`.

The rest is an application of the functions `heron` and `trapez`:

`√(heron(4.8, 7.2, 6.6) + trapez(12.1, 7.2, 6, s) + 5.4*s/2 + heron(5.4, 7.4, 9.8) + 4.9^2*√(3)/2)`, result: 12.028 km.

d) `solve(heron(1, 2, x)=10, x)`. The result given is false.

Domain `[1, 3]` (read from the graph of `heron(1, 2, x)` using the trace feature).

Choose \overline{AB} as the side with length 1 of the triangle and \overline{AC} with length 2. Then the vertices C are on a circle with radius 2 and centre A .

For all circles with radius x and centre B the area will be 0 if the circles are touching. This is for the cases $x = 1$ (smallest possible value) and $x = 3$ (largest possible value).

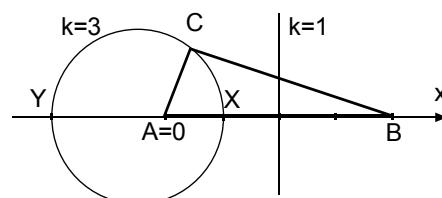
The triangle has the largest height and hence the largest area if side \overline{AC} is perpendicular to \overline{AB} , i.e. $A = 1$. To illustrate the situation dynamically you can do the construction with Cabri.

- e) The locus is the Apollonian circle over the side with length 4 and ratio 1: k . You find the diameter of the circle at area 0 by entering `solve(heron(4, x, k*x)=0, x)` and this gives the result

$$x = \pm \frac{4}{k+1} \text{ and } x = \pm \frac{4}{k-1}, \text{ where } \frac{4}{k+1} \text{ corresponds to}$$

the inner intersection point X and $-\frac{4}{k-1}$ to the outer

intersection point Y . For discussion use an x -axis through A and B with $A = 0$.



For $k = 3$ the x -values are 1 and -2, i.e. the Apollonian circle has radius 1.5 and the largest area is therefore 3.

For an arbitrary k the radius is half of the difference of the above values: $\frac{4k}{(k-1) \cdot (k+1)}$.

For $k \rightarrow \infty$ (or $k = 0$) the Apollonian circle shrinks to point A (or B). For $k = 1$ you get an isosceles triangle with sides 4, x and x .

These ideas will challenge and expand students' minds as they consider the geometry associated with the limit $k \rightarrow 1$ of the Apollonian circle with radius and centre point tending to "infinity". The extension to the case $0 \leq k \leq 1$ may be difficult for students to understand geometrically. However, it is useful to extend students' thinking by getting them to consider what happens near $k=1$ with the circle and its centre by "passing through infinity" from $-\infty$ to $+\infty$.

5.4 Ages 15-17 – Exploring the binomial theorem

This nice exploration can be done using either DERIVE or one of the TI-CAS family. This is the TI-worksheet. It can easily be adapted for other Computer Algebra systems. You will find very few system-specific notes. The students know how to handle the very few CAS commands given in the worksheet.

In this activity we want to calculate - or **expand** - higher powers of binomials, such as $(4x^2y - 3xy^3)^8$ How can we do this with your CAS?

First enter the expression, then simply “expand“ it.

The result is impressive! What is the calculation time?

Write down the 1st term of the expansion:

Write down the 4th term: and the last term:

Now use your system to expand the powers of $(a + b)$ and note only the coefficients of each term. (Write down the numbers neatly, one beneath the other.)

$$n = 4: (a + b)^4:$$

$$n = 5: (a + b)^5:$$

$$n = 6: (a + b)^6:$$

$$n = 7: (a + b)^7:$$

If you can recognize a pattern in your results, then add two more lines (for $n = 8$ and 9) without using your CAS. Then check the coefficients by using the CAS. If you do not see a pattern yet, then don't worry, let your device do the work.

Complete the pattern for $n = 3, 2, 1$

How should the first line (for $n = 0$) look?

What can you conclude from this line? Check it!

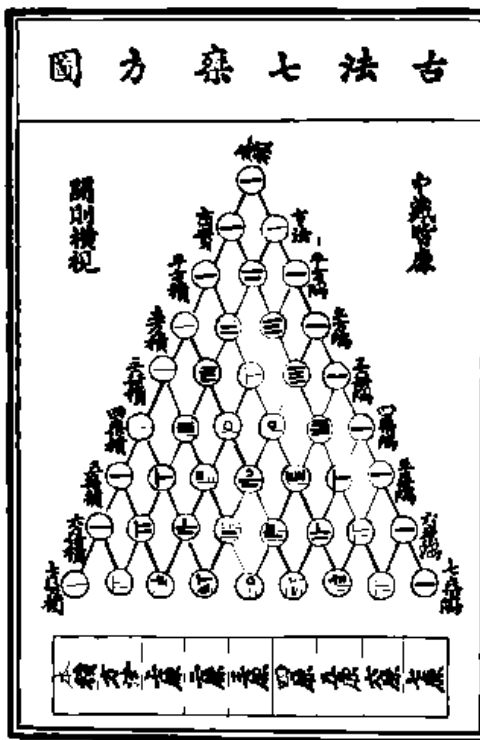
.....

Write down the numbers from the coefficients' pattern in the form of a triangle:

- (n = 0) 1
- (n = 1) 1 1
- (n = 2) 1 2 1
- (n = 3)
- (n = 4)
- (n = 5)
- (n = 6)
- (n = 7)
- (n = 8)
- (n = 9)
- (n = 10)
- (n = 11)

Can you explain an obvious pattern?

We call this triangle of numbers **Pascal's Triangle** (Blaise Pascal, 1623-1662; but the triangle on the left is found in a Chinese paper from 1303 a.d.).



The other picture is from an old 16th century German book for merchants.

Both pictures are from the - highly recommended - book *AGNESI to ZERO*, Key Curriculum Press.

Now we are going to investigate some properties of the triangle.

Find the sum of the numbers in each line! (You can note the sum on the end of the lines.)

What do you notice?

Are you able to prove this fact? (Hint: Substitute for a and b !)
.....

The students should know the “coefficient proof” for quick checking of their results for calculating with general expressions, e.g. $(3a^2 + 4ab)(ab - 2a + 2b^2) = \dots\dots\dots$, by comparison of $(3 + 4)(1 - 2 + 2) = 7$ with the sum of the coefficients of the result.

Observe the powers of a and b in the partial expressions of the expansions of the first five powers of $(a + b)$.

Try to expand $(a + b)^8$ by-hand using PASCAL’s Triangle.

$(a + b)^8 = \dots\dots\dots$

Check your result using the device.

Try once more with $(2x + 3y)^5$ without using CAS.

$(2x + 3y)^5 = \dots\dots\dots$

Expand all the partial expressions by-hand:
.....

Check your results using CAS:

Expand $(a + b)^5$, then replace (substitute) a and b by the expressions $2x$ and $3y$ using the [1]-operator:

Compare the result offered by the CAS with your one!

Expanding once more if necessary will lead to a final result.

You can now expand $(2x + 3y)^5$ directly without intermediate substitution.

Now try by-hand: $(4z + 2u)^6 = \dots\dots\dots$
 $= \dots\dots\dots$

Compare your result with the CAS-result!

Explain the effect of the changed sign in $(a - b)^n$ in your own words.
.....

Exercises

Write down the following expansion without the machine and then check your results:

$$(z + 4)^6 = \dots\dots\dots$$
$$= \dots\dots\dots$$

Repeat the of exponents first (product and power of powers) and then proceed with the next problem.

$$(4x^2y - 3xy^3)^8 = \dots\dots\dots$$
$$= \dots\dots\dots$$

$$(5ab + 2b^2)^9 \quad \text{the 3rd term: } \dots\dots\dots \quad \text{the 7th term: } \dots\dots\dots$$

$$(2u + 1.5a)^5 = \dots\dots\dots$$

Expand the expression $(3a^2 - 2ab + b^2)^5$ without using the `expand`-command of your Computer Algebra System.

Describe how you tackled this problem.

.....

Imagine that you have written down the result of the expansion, sorted with respect to descending powers of variable a . What are the first three terms?

Compare these with the result from the TI .

The rule or law for calculating the powers of binomials is called **the Binomial Law**.

The numbers appearing in Pascal's triangle and their special order have fascinated mathematicians for a very long time, and they still do. Use the sheet on the next page to experiment with the numbers in this famous triangle.

Mark all even numbers using a felt pen. First make one or two copies for suggested investigations.

Do you observe a special pattern?

Suggested investigations:

- Mark all numbers which are divisible by 3
- Mark all numbers which leave the remainder 2 after division by 4
- Design your own investigation of a similar kind.

We can recognize so called "self similarity" which is a property investigated in the very modern "Chaostheory".

Using the two programs `pas()` and `invpas()` you can generate various "pictures" emerging from Pascal's Triangle.

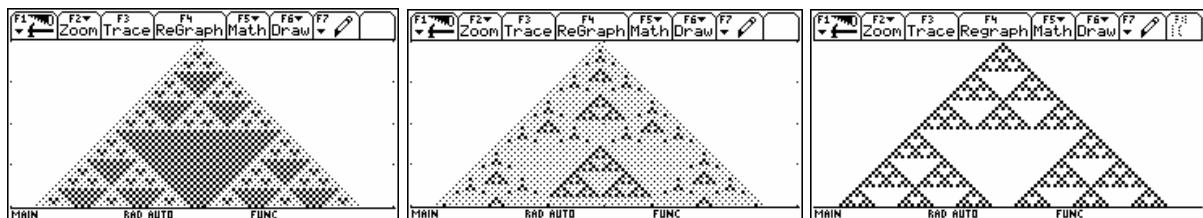
`pas(n,divisor,remainder)` marks all locations in the first n rows of Pascal's Triangle. where the division of the number by `divisor` leaves the `remainder`.

`invpas(n,divisor,remainder)` returns a black spot at these positions, where we do **not** receive the remainder.

To mark the even coefficients in the first 30 rows then one has to enter `pas(30,2,0)`. Explain the function arguments.

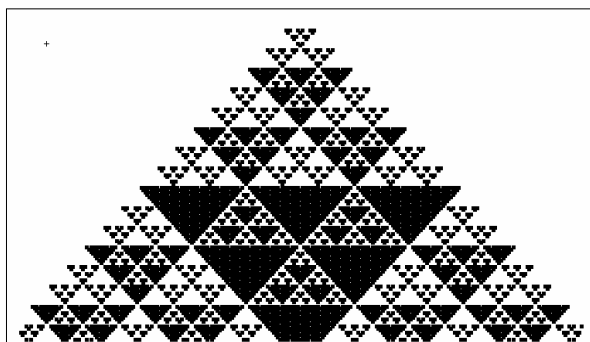
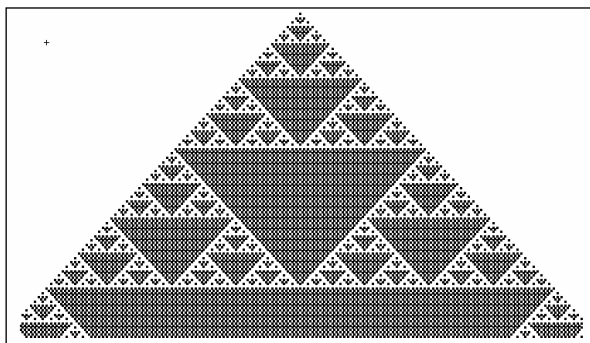
Try to reproduce your hand made paintings on the TI screen. How many rows can be represented on this small screen?

Which parameters in `pas()` generate the following pictures?



Try to produce two "extra nice" Pascal pictures. We will collect the triangles for a *PASCAL*-Poster.

Here are three *DERIVE*-plots showing the even numbers, the numbers which are divisible by nine and the remainders after division by 5.



```

      [ 1 ]
      [ 1, 1 ]
      [ 1, 2, 1 ]
      [ 1, 3, 3, 1 ]
      [ 1, 4, 1, 4, 1 ]
      [ 1, 0, 0, 0, 0, 1 ]
      [ 1, 1, 0, 0, 0, 1, 1 ]
      [ 1, 2, 1, 0, 0, 1, 2, 1 ]
      [ 1, 3, 3, 1, 0, 1, 3, 3, 1 ]
      [ 1, 4, 1, 4, 1, 1, 4, 1, 4, 1 ]
      [ 1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 1 ]
      [ 1, 1, 0, 0, 0, 2, 2, 0, 0, 0, 1, 1 ]
      [ 1, 2, 1, 0, 0, 2, 4, 2, 0, 0, 1, 2, 1 ]
      [ 1, 3, 3, 1, 0, 2, 1, 1, 2, 0, 1, 3, 3, 1 ]
      [ 1, 4, 1, 4, 1, 2, 3, 2, 3, 2, 1, 4, 1, 4, 1 ]
    
```

The program code for pas(n,a,r):

```

pas(n,a,r)
Prgm
ClrDraw
FnOff :PlotsOff
setGraph("Axes","Off")
setGraph("Grid","Off")
Local i,j,z,pt
Define pt(x,y)=Prgm
Px10n x,y:Px10n x,y:Px10n x+1,y
Px10n x,y-1:Px10n x+1,y-1
EndPrgm
For i,0,n
  For j,0,i
    nCr(i,j)→z
    If mod(z,a)=r Then
      pt(1+2*i,117-2*i+4*j)
    Else
      Px10n 1+2*i,116-2*i+4*j
    EndIf
  EndFor
EndFor
EndPrgm

```

Two explanations:

nCr(i,j) generates the *binomial coefficients* using a rule which you will learn later in probability theory.

Function mod(z,a), *Modulo Function*, returns the integer remainder after division of the integer z by the integer a.

Find the following using mental computation then using your device:

mod(87,9) =

mod(15,3) =

mod(113,7) =

mod(200,13) =

Which line of the program above would you will have to change to change pas() to invpas(), to return the *inverse* figures?

Assessment example (C3)

Given are some terms of a binomial expansion. Complete the unknown terms

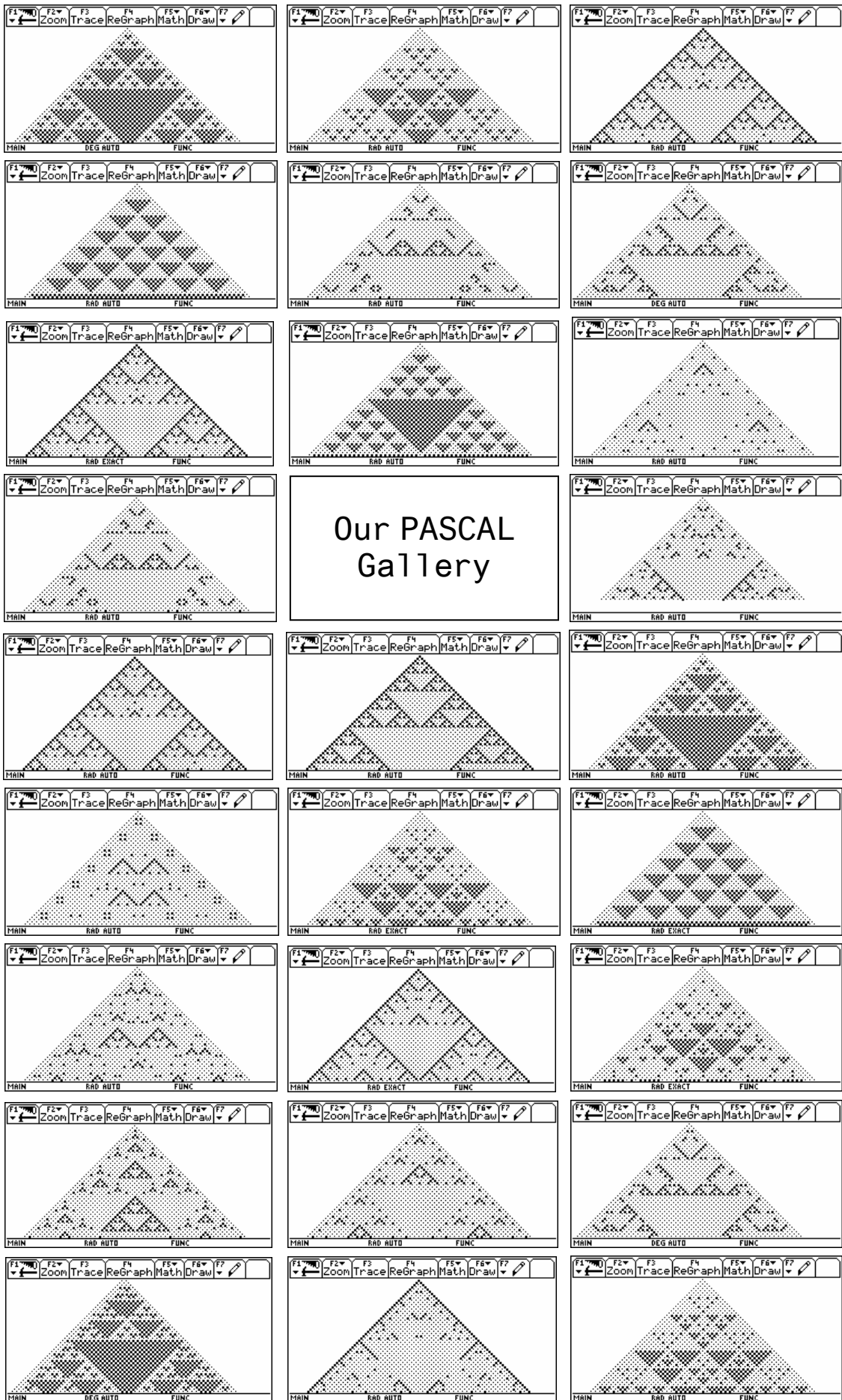
a) $x^4 - 16x^3 + 96x^2 - \dots$

b) $a^6b^3 + 6a^4b^5 \dots = (\dots)^3$

c) $\dots - 180x^3z \dots = (\dots)^2$ Give more than one solution!

d) $729x^{12} - 5832x^{10}y + 19440x^8y^2 \dots =$

On the next page we can see the results of the efforts to create a fine poster.



5.5 Ages 15-17 – Trivialisation and dynamisation with examples from elementary vector geometry

Classical advanced-level exercises are often considered difficult because their solution normally requires extensive calculations. The effort to develop suitable exercises, which can be solved in a reasonable amount of time, is usually considerable. Two examples from elementary vector geometry (distance calculation and parametric representation of straight lines) will illustrate how classic exercises can be trivialised using CAS.

Furthermore, these examples demonstrate that such exercises can be extended to become dynamic and more demanding tasks by parameterisation. For each example some similar exercises are given as activities for students. These can be used both as examination questions and for training purposes.

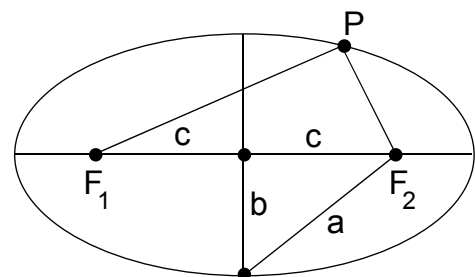
The examples show that mathematics does not become trivial even though some exercises may be trivialised. Rewriting examples to include parameters means that students are required to know even more mathematics than they would need in a non-CAS situation.

5.5.1 Example 1: Construction of an ellipse with a thread (category C1)

Definition

The ellipse is the locus of all points P , where the sum $s = 2a$ of the distances from P to two fixed points F_1 and F_2 (focus) is a constant (length of the thread).

Let $2c = |F_1F_2|$ be the distance of the two foci (see figure), then a is the semi-major axis of the ellipse and $b = \sqrt{a^2 - c^2}$ is the semi-minor axis.



- Use Cabri Geometry to draw the ellipse with the above definition. The basic objects are the foci and the length of the string $s = 2a$.
- Determine the equation of the ellipse. Choose foci $F_1 = (-c; 0)$, $F_2 = (c; 0)$ and the point on the ellipse as $P = (x, y)$. Set up the condition in accordance with the above definition. Transform the ellipse equation into the following form (use expand to simplify and don't forget to replace c):

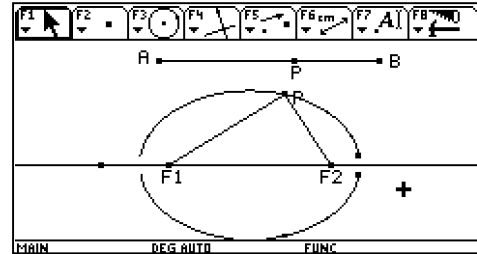
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Solve the ellipse equation for y and draw the two functions for $a = 4$ and $b = 3$.
- Show that $x(t) = a \cos(t)$ and $y(t) = b \sin(t)$ satisfy the ellipse equation. Draw the ellipse in parametric representation for $a = 4$ and $b = 3$ and compare with c).

Solutions to example 1

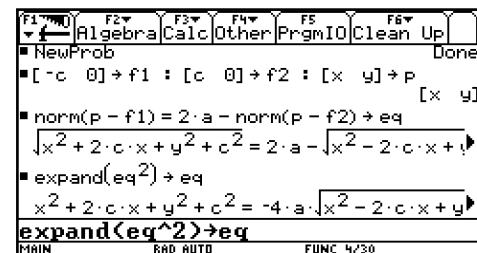
First have students carry out this construction by-hand (working in pairs). To do this one merely needs a piece of thread (or string). One student holds the thread ends at the foci and the other student places the pencil at point P , drawing the ellipse by keeping the thread tight. The students can experiment freely or draw ellipses with given values for a and c .

- a) Possible solution: Construct a segment \overline{AB} (length of the thread) and a point P on it. Construct a circle with centre F_1 and radius $|AP|$ and another circle with centre F_2 and radius $|BP|$ (transmission of the segment). One of the intersection points is P . Draw the locus of both intersection points, while the corresponding point P moves along the segment \overline{AB} .

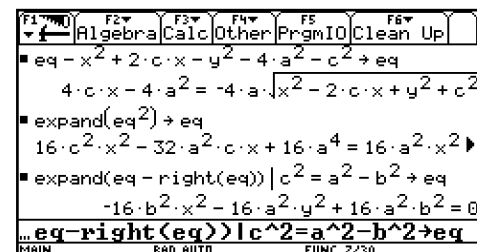


The ellipse will not be closed entirely, even if the maximum number of points is chosen. With the PC-version of Cabri you will obtain the result faster and the ellipse will be more beautifully formed.

- b) Find a possible solution from the screenshot to the right. First you define the points. Then store the condition (equation eq) in such a way that by squaring it, a term as simple as possible arises. The next step: square and expand the equation and isolate the root on the right side.



Squaring once more and expanding delivers an equation without roots (the new equations are stored in eq at each step).



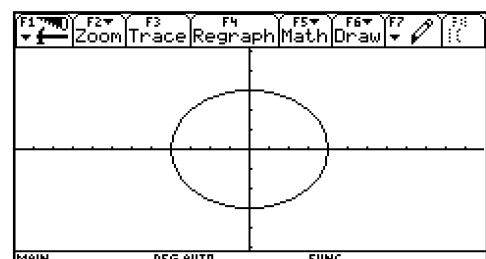
Using $eq - right(eq)$ bring all terms to one side and in the same step substitute for c^2 .

You will obtain the desired equation by dividing through by $-16a^2b^2$ and expanding.

The whole calculation can also be done by-hand. Use of the computer however shortens the computing time considerably.

- c) Result: $y = \pm \frac{b\sqrt{a^2 - x^2}}{a}$.

Store the functions in $y1(x)$ and $y2(x)$ and draw the two halves of the ellipse. The ellipse is drawn completely if the points on the x -axis are ellipse points. You achieve this with $ZoomDec$ and $xres=1$ (window settings).



$$d) \frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} = \cos(t)^2 + \sin(t)^2 = 1$$

Switch function mode to parametric: **MODE** Function/Parametric, and angle to radian: **MODE** Angle/Radian. Draw the graph and you get the same figure as in c) (adapt window settings if necessary).

Note: Drawing on previous knowledge, you can also treat transformations in the coordinate system or use dilation e.g.: Translation by 3 in x -direction and 4 in y -direction.

$$\frac{(x-3)^2}{a^2} + \frac{(y-4)^2}{b^2} = 1.$$

The ellipse arises from the unit circle $x^2 + y^2 = 1$ by dilation in respect to the x - and y -axis with factors a and b .

Another possibility is to introduce the equations of a hyperbola and parabola in an analogous way.

5.5.2 Exercises for example 1 - practise or examination, all of category C3

Store the ellipse equation as a function $e1(x, y, a, b)$.

- (1) Consider ellipses with centre $O = (0; 0)$ and semi-axes 5 and 3 on the x - and y -axis
 - a) (classical) Does point $(4; 2)$ lie on the ellipse?
 - b) What are the conditions for c , so that point $(c; 2)$ lies on (within) the ellipse?
 - c) Determine c , so that point $(c; 4)$ lies on the ellipse.
 - d) Determine the semi-axes a and b and the foci of an ellipse, which goes through the points $(4; 3)$ and $(2; -5)$.

- (2) Consider ellipses with centre O and semi-axes 5 and 3 on the x - and y -axis.
 - a) (classical) Find the intersection of the ellipse and the straight line $y = 2x + 1$.
 - b) For which values of c is $y = c \cdot x + 1$ a tangent to the ellipse?
 - c) For which values of a is $y = x + c$ a tangent to the ellipse?
 - d) Determine an ellipse with ratio 3:1 for the axes which touches the line $y = 2x + 1$ (centre O , axes of the ellipse on the x - and y -axis).
 - e) For which values of c does $y = c \cdot x + 1$ cut a chord with length 8 of the given ellipse?

- (3) Given: $A = (1; 0)$ and $B = (4; 0)$.
 - a) (classical) For which points P on the straight line $y = x + 1$ is the distance to point B twice the distance to point A ?
 - b) Determine the locus of all points whose distance to point B is twice the distance to point A . (equation and geometrical interpretation.)
 - c) Complete the same task as for b), but this time let $B = (c; 0)$.
 - d) Consider the same situation as in b) for the locus of all points with ratio $|BP| : |AP| = k$. What kind of locus is this? What are the intersection points of this locus with the x -axis? Where do these points tend to, if k tends to 1? Interpret the result geometrically.

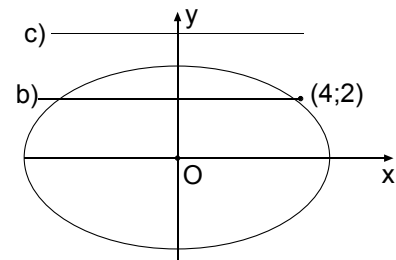
- (4) Given: $A = (-12; -3)$, $B = (6; 9)$, $C = (13; 2)$
- (classical) Determine the centre M and the radius r of a circle which goes through points A , B and C .
 - Determine the equation of the circle found in a).
 - Solve task a) with $C = (13; k)$ and discuss the possible cases (geometrical interpretation).

Solutions to exercises 1 to 4

S1 a) `e1(4, 2, 5, 3)` Result: false.
The point does not lie on the ellipse.

b) `solve(e1(c, 2, 5, 3), c)`
Result: $c = \pm \frac{5\sqrt{5}}{3}$ ($-\frac{5\sqrt{5}}{3} < c < \frac{5\sqrt{5}}{3}$)

c) `solve(e1(c, 4, 5, 3), c)` Result: false.
It may be that the major axis lies on the y -axis.
Therefore also test: `solve(e1(c, 4, 3, 5), c)`
Result: $c = \frac{9}{5}$ or $c = -\frac{9}{5}$.



d) `solve(e1(4, 3, a, b) and e1(2, -5, a, b), {a, b})`

Result: $a = \frac{\sqrt{91}}{2} \sim 4.770$, $b = \frac{\sqrt{273}}{3} \sim 5.508$, i.e. the major axis points in the direction of the y -axis; therefore, c can be calculated with $c = \sqrt{a^2 - b^2} = \frac{\sqrt{273}}{6} \sim 2.754$; this results in the foci $F_{1,2} = \left(0; \pm \frac{\sqrt{273}}{6}\right)$.

The above tasks could also be solved by-hand, but the effort of solving one exercise is approximately as great as solving all the tasks with CAS. The main emphasis therefore shifts from calculating to analysing mathematical expressions.

S2 a) `solve(e1(x, 2x+1, 5, 3), x)` Result: $P_1 = (0.971; 2.943)$, $P_2 = (-1.889; -2.778)$.

b) The straight line intersects the y -axis at $(0; 1)$, therefore you always have two intersection points. Thus a tangent is impossible. You can also solve this by calculating the intersection points as in a). The discriminant $25c^2+8$ is always positive.

c) `solve(e1(x, x+c, 5, 3), x)` Result: $x_{1,2} = \frac{5 \cdot (-5c \pm 3\sqrt{34 - c^2})}{34}$.

With the double solution, i.e. $c_{1,2} = \pm\sqrt{34}$, you get the tangent.

d) `solve(e1(x, 2x+1, a, a/3), x)` Result: $x = \frac{-18 \pm \sqrt{37a^2 - 9}}{37}$.

Hence the lengths of the semi-axes are $a = \frac{3\sqrt{37}}{37}$ and $b = \frac{\sqrt{37}}{37}$.

Don't forget the other solution with

$$\text{solve}(e1(x, 2x+1, a/3, a), x) \quad \text{Result: } x = \frac{-2 \pm \sqrt{13a^2 - 9}}{13}$$

and we get an ellipse with $a = \frac{3\sqrt{13}}{13}$ and $b = \frac{3\sqrt{13}}{13}$.

e) $\text{solve}(e1(x, c*x+1, 5.3), x)$ Result: $x_{1,2} = \frac{5 \cdot (-5c \pm 3\sqrt{25c^2 + 8})}{25c^2 + 9}$

Difference of x-values: $x_2 - x_1 = \frac{30\sqrt{25c^2 + 8}}{25c^2 + 9}$ (store it in d).

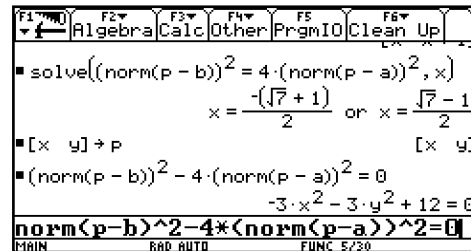
Difference of y-values: $y_2 - y_1 = c \cdot (x_2 - x_1) = c \cdot d$.

Distance: $\text{solve}(d\sqrt{(1+c^2)}=8, c)$ Result: $c_{1,2} = \pm 0.605$.

S3 a) First store the points A, B, P :

$[1, 0]$ $\text{STO} \rightarrow$ a, $[4, 0]$ $\text{STO} \rightarrow$ b, $[x, x+1]$ $\text{STO} \rightarrow$ p.

See solutions in the screenshot to the right.



b) With $P = (x; y)$ you get the equation of the circle

$x^2 + y^2 = 4$ (divide the last equation by -3), that is, the circle with centre at the origin and radius 2. This is the Apollonian circle over AB with ratio 2.

c) $[c, 0]$ $\text{STO} \rightarrow$ b and dissolution as in b) gives the equation $x^2 + \frac{2 \cdot (c-4) \cdot x}{3} + y^2 - \frac{c^2 - 4}{3} = 0$.

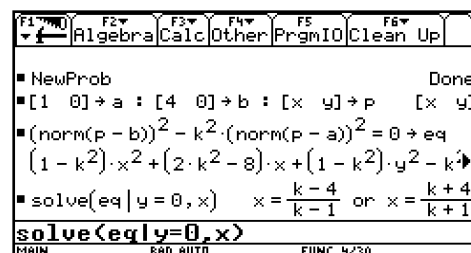
The equation can be transformed to standard form by completing the square. This only works by-hand, which shows that students still need to be able to perform simple algebraic

manipulations by-hand: $\left(x - \frac{4-c}{3}\right)^2 + y^2 = \frac{4c^2 - 8c + 4}{9}$.

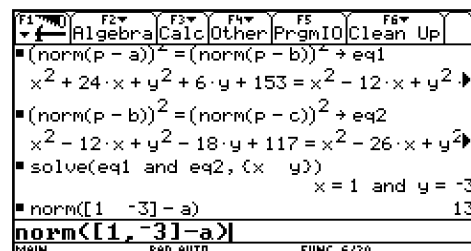
This is a circle with centre $M = \left(\frac{4-c}{3}; 0\right)$. The right side is the square of the radius.

For example with $c = 100$ the centre is $(32; 0)$ and the radius is approximately 48.069.

d) The two solutions behave differently for $k \rightarrow 1$ (see accompanying screenshot). The second value tends to 2.5 (inner point to the ratio k), i.e. to the centre of the segment \overline{AB} . The first value tends to $\pm\infty$. This corresponds to a circle with an “infinitely” large diameter, i.e. the perpendicular bisector of \overline{AB} .



S4 a) First store the points A, B, C and $P = (x; y)$. The conditions $|AP| = |BP|$ and $|BP| = |CP|$ correspond to the two perpendicular bisectors of \overline{AB} and \overline{BC} . Their intersection point is the centre $M = (1; -3)$ of the circle we found. The radius is $r = |MA| = 13$.



b) $|AP| = 13$

With CAS: $\text{norm}(p-a)^2 = 13^2$

Result: $x^2 + 24x + y^2 + 6y + 153 = 169$.

Complete the squares to give:

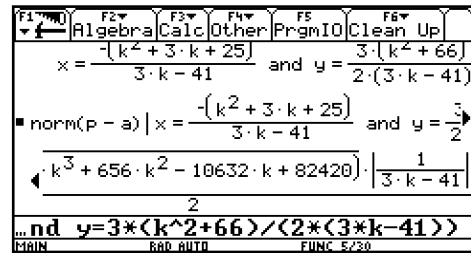
$(x + 12)^2 + (y + 3)^2 = 169 = 13^2$.

c) Replace C by $(13; k)$ and carry out the same steps as in the screenshot of a).

Result: $x = \frac{-(k^2 + 3k + 25)}{3k - 41}, y = \frac{3 \cdot (k^2 + 66)}{2 \cdot (3k - 41)}$

If k tends to $\frac{41}{3}$ then x, y and r tend to ∞

(r see screenshot).



Interpretation: A, B and $C = \left(13; \frac{41}{3}\right)$ lie on a straight line.

5.5.3 Example 2: Elementary vector geometry in space (Category C3)

Given the points $A = (4; -5; 11), B = (1; -\frac{1}{2}; 2)$ and the straight line $g: \vec{r} = \begin{pmatrix} 0 \\ -13 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$.

- (classical) Determine all points P on the straight line g with ratio $|AP| : |BP| = 2:1$.
- Replace the third component 1 in the direction vector of the straight line g with c (in this case the direction of the straight line becomes variable). Which values of the parameter c results in one, two or no solution?
- On which locus do these intersection points lie? (points with $|AP| : |BP| = 2:1$, geometrical reasoning).
- Determine the characteristic quantities of this locus.
- We now omit the condition $P \in g$, i.e. $P = (x; y; z)$. Determine the equation for x, y, z of the locus in d). Show that it is the equation of a sphere ($|MP|^2 = r^2$) applying d).

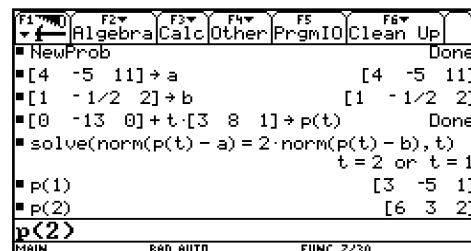
Solutions to example 2

a) The problem can be described by the following conditions:

I. $\frac{|\vec{AP}|}{|\vec{BP}|} = 2$ or $|\vec{AP}| = 2 \cdot |\vec{BP}|$ II. $P \in g$

Solution (see screenshot): First store the points (= position vectors) in a and b and the straight line in $p(t)$. Thus equation II is taken into account. Solve equation I according to the screenshot.

Result: $P_1 = (3; -5; 1)$ and $P_2 = (6; 3; 2)$.



These simple results show that the exercise was prepared with much effort to reduce the amount of calculation by-hand. With CAS you can take any values without augmenting the calculation time. Thus it is much easier for the teacher to prepare exercises.

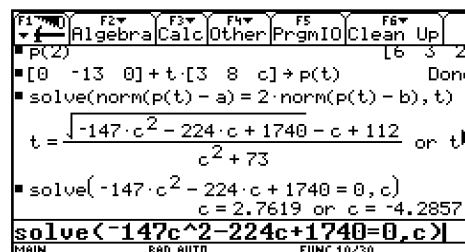
After setting up the conditions, the remaining task consists of rearranging the terms and solving a quadratic equation. CAS calculators trivialise this part regardless of the complexity of the result.

- b) With CAS you just change $p(t)$ and retrieve the equation from history. The equation was squared in order to accelerate calculations:

$$\text{Result: } t_{1,2} = \frac{-c + 112 \pm \sqrt{-147c^2 - 224c + 1740}}{c^2 + 73}$$

The solutions are determined by the discriminant:

- One solution: $c = -\frac{30}{7} = -4.286$ and $c = \frac{58}{21} = 2.762$,
- Two solutions: $-4.286 < c < 2.762$,
- No solution: $c < -4.286$ or $c > 2.762$.



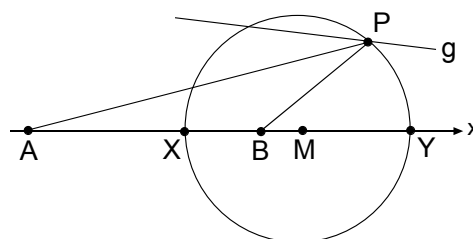
With a single discussion of such a solution students will most likely learn more than by solving several drill exercises of type a).

- c) The points lie on the Apollonian sphere over the segment \overline{AB} with ratio 2:1.
- d) We determine the points X, Y on the Apollonian sphere, which lie on the straight line AB (diameter of the sphere).

$$\overrightarrow{OX} = \overrightarrow{OA} + \frac{2}{3} \overrightarrow{AB}, \quad \overrightarrow{OY} = \overrightarrow{OB} + \overrightarrow{AB},$$

$$\overrightarrow{OM} = \frac{1}{2} (\overrightarrow{OX} + \overrightarrow{OY})$$

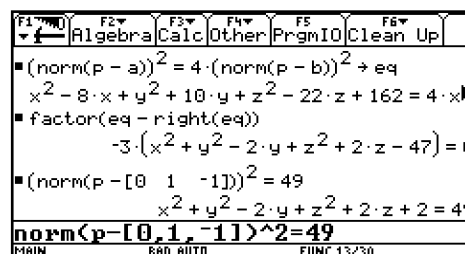
Result: $X = (2; -2; 5), Y = (-2; 4; -7),$



- e) $[x, y, z] \overline{\text{STO}} p$. For further steps see the screenshot on the right. Depending on interest and previous knowledge the equation can be brought to the standard form by completing the square:

$$x^2 + (y - 1)^2 + (z + 1)^2 = 49 \Leftrightarrow |MP| = r^2$$

(Translation of the sphere $x^2 + y^2 + z^2 = 49$)



5.5.4 Exercises for example 2 - practise or examination, all of category 3

The philosophy of all tasks is to replace numbers in example 2 by parameters and create new situations, which help to promote spatial imaginative power. The fact that CAS results are extensive forces a more abstract way of thinking which concentrates on the essentials of expressions.

- (5) Given: $A = (4; -5; 11)$, $B = (1; -\frac{1}{2}; 2)$ and the straight line g (see example 1).
- Let $c = 1$. Determine the points P on the straight line g , for which $|AP| : |BP| = 2:1$.
 - Let c be arbitrary. For which values of the parameter c do solutions exist?
 - Interpret the result of a) and b) and compare results with those obtained for example 2.

(6) Given: A and B as in exercise 5 and $g: \vec{r} = \begin{pmatrix} c \\ 1 \\ d \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$.

- Let $c = 0$ and $d = 0$. Determine the points on the straight line g , for which $|AP| : |BP| = 3:1$.
- Determine the radius r of the Apollonian sphere over \overline{AB} with ratio 3:1
- Let $c = 10$ and $d = 0$. Complete the same task as for a).
- Let $c = 10$ and d be arbitrary. Complete the same task as for a).
- Interpret the results of b) and c).

(7) Consider A and B as in exercise 5 and the straight line $g: \vec{r} = \begin{pmatrix} 0 \\ -13 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$ (see example 2).

Let us analyse the points $P \in g$, for which $|AP| : |BP| = k:1$ ($k > 0$).

- Find a formula for t where k is arbitrary.
- The formula fails for a certain k . Discuss why this is the case.
- For which other values do solutions exist?
- Determine the radius of the Apollonian sphere r using the intersection points with AB .
What happens as k tends to 1?

Solutions to exercises 5 to 7

S5 a) Result: $t_{1,2} = \frac{\pm 7\sqrt{74}}{74}$, $P_1 = (2.41; 7.510; -0.186)$, $P_2 = (-2.441; -5.510; -1.841)$

b) Result: $t_{1,2} = \frac{\pm 7}{\sqrt{c^2 + 73}}$, i.e. there are always solutions.

- c) The solutions are always symmetrical to the starting point of g .

Interpretation: The starting point of g is the centre of the Apollonian sphere (compare with example 2). Therefore, every value of c results in two intersection points.

To further improve students spatial imagination ask them whether there are other straight lines with two intersection points for every c (symmetric to the starting point or not).

You will have the general case if the starting point S of g lies within the Apollonian sphere and the symmetric case if, in addition, g is perpendicular to SM (M = centre of the sphere).

S6 a) Compared to example 2 only the ratio of the distances changes.

Result: $t_1 = -0.496$ and $t_2 = 0.367$, $P_1 = (-1.487; -2.965; -0.496)$ and $P_2 = (1.102; 3.938; 0.367)$.

b) In analogy with example 2d) we calculate the intersection points of the Apollonian sphere with AB :

$$\vec{OX} = \vec{OA} + \frac{3}{4} \vec{AB},$$

$$\vec{OY} = \vec{OB} + \frac{1}{2} \vec{AB},$$

$$r = \frac{|\vec{XY}|}{2} = \frac{63}{16} = 3.938.$$

c) Result: false. The straight line doesn't intersect the Apollonian circle.

d) Here we get a formula for t : $t = -0.115 \cdot (0.117 \cdot (d + 34.750) \pm \sqrt{-d^2 + 2.726 \cdot d - 58.503})$ but the discriminant is always negative (CAS doesn't give you this important hint), i.e. there is no solution.

Geometric Interpretation: The plane in which the straight line moves as d is changed doesn't intersect the Apollonian sphere.

S7 a) Replace the ratio 2 by k : `solve(norm(p(t)-a)^2=k^2*norm(p(t)-b)^2,t)` gives

$$t = \frac{6 \cdot (35 \cdot k^2 - 29) \pm \sqrt{-6 \cdot (605 \cdot k^4 - 5691 \cdot k^2 + 4870)}}{148 \cdot (k^2 - 1)}.$$

b) For $k = 1$ you get a division by 0. If you solve the above equation with $k = 1$ the result is

$$t = -\frac{53}{48}. \text{ Geometric interpretation:}$$

The perpendicular bisector of \overline{AB} intersects g at a point.

c) The discriminant in a) determines the solutions. They lie in the range $0.976 \leq k \leq 2.908$.

d) First store an arbitrary position vector $[x, y, z]$ `STO>` p and the condition $|AP| : |BP| = k:1$ (Apollonian sphere) in the form `norm(p-a)^2=k^2*norm(p-b)^2` `STO>` eq .

Store the parametric equation of the straight line through A and B (the position vectors to A and B are still stored in a and b): `a+t*(b-a)` `STO>` $s(t)$.

Substitute x, y and z in eq with the components of s :

$$eq|x=s(t)[1,1] \text{ and } y=s(t)[1,2] \text{ and } z=s(t)[1,3].$$

Solving this equation with respect to t gives: $t_1 = \frac{k}{k+1}$ or $t_2 = \frac{k}{k-1}$ (not dependent on A and B !), whereas t_1 belongs to the inner intersection point X and t_2 to the outer one Y .

You get the centre of the Apollonian circle with $\frac{1}{2}(s(t_1) + s(t_2))$ or $s(\frac{1}{2}(t_1 + t_2))$.

$$\text{Result: } M = \left(\frac{3}{2 \cdot (k+1)} - \frac{3}{2 \cdot (k-1)} + 1; \frac{-9}{4 \cdot (k+1)} + \frac{9}{4(k-1)} - \frac{1}{2}; \frac{9}{2 \cdot (k+1)} - \frac{9}{2 \cdot (k-1)} + 2 \right)$$

$$\text{and the radius with } r = \frac{|XM|}{2}, \text{ result: } \frac{21}{2} \left| \frac{k}{(k-1) \cdot (k+1)} \right|.$$

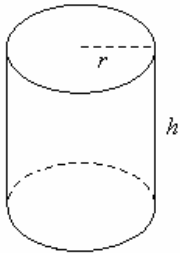
*Of course such a long formula has to be tested, e.g. with $k \rightarrow 0, 1, \infty, 3$ (Exercise 6 b).
Such discussions deal with limits in an intuitive way without any theory of limits.*

Note: A great number of classical exercises, which are trivialised by CAS, can be expanded to become demanding tasks by introducing parameters. Some of the examples above show that with such new tasks you will often come across subjects, which are usually dealt with later. I used this teaching sequence in the 10th school year as an application of vectors with coordinates.

5.6 Ages 17-19 – Optimisation

5.6.1 A standard example

As an introduction, do a)-h) without using CAS



The cylinder shown has the volume $1.0 \text{ litre} = 1 \text{ dm}^3$.

We wish to determine the radius r and height h that gives the minimum surface area of the cylinder.

- Write the volume $V = 1 \text{ dm}^3$ of the cylinder in terms of h and r .
 - Solve the equation in (1) for h .
 - Write the surface area A of the cylinder (side+bottom+top) in terms of h and r .
 - Substituting h from (3), write A as a function of r .
 - Determine the derivative $A'(r)$.
 - Solve the equation $A'(r) = 0$ for r .
 - Calculate the corresponding value of h by substituting the value of r just found into (3).
- Note: The unit of h and r is dm.
- Compare the numerical values of h and r . Conjecture? Prove the conjecture.

Solution (partial)

The numerical values found for the special case $V = 1.0 \text{ litre}$ suggest the conjecture that the minimum area is obtained when height = diameter. Of course it is possible to prove the conjecture algebraically using paper and pencil methods only. Below we show how to do it with CAS.

$$\frac{V}{\pi \cdot r^2} \rightarrow h$$

$$2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h \rightarrow a(r)$$

$$\text{solve}\left(\frac{d}{dr}(a(r)) = 0, r\right) \quad r = \frac{2^{2/3} \cdot V^{1/3}}{2 \cdot \pi^{1/3}}$$

$$\text{solve}(d(a(r)), r) = 0, r$$

$$h = 2 \cdot r \mid r = \frac{2^{2/3} \cdot V^{1/3}}{2 \cdot \pi^{1/3}} \quad \text{true}$$

$$\frac{2^{2/3} \cdot V^{1/3}}{2 \cdot \pi^{1/3}} \mid r = \frac{2^{2/3} \cdot V^{1/3}}{2 \cdot \pi^{1/3}}$$

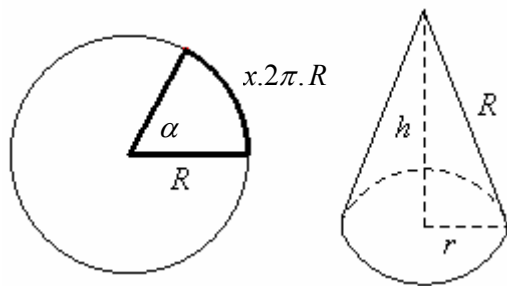
Exercise

Consider a similar problem for a cylinder without a top.

- Determine the dimensions giving the minimum surface area for a volume of 1.0 litre .
- Compare height and radius for any volume V where the cylinder has a minimum surface area.

Answer to b: height = radius.

5.6.2 Make a cone



A section is cut from a circular piece of paper, and folded to make a cone. The radius of the paper is R . You now remove a fraction x of the paper. If for example $x = 0.25$, the centre angle α of the section cut out for the cone is 90° . The base radius of the cone is given by r and the height by h .

On a Voyage 200 we can write rr for R , to distinguish between the base radius of the cone and the radius of the circular piece of paper from which the cone is made.

- Determine r and h in terms of x and R and find the volume $V(x)$ of the cone in terms of x and R
- Determine $V'(x)$ and solve the equation $V'(x) = 0$.
- Calculate the fraction x and the corresponding angle α that gives the maximum cone volume
- Repeat this problem with the centre angle α as the independent variable.

Solution (partial)

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
solve(x * 2 * pi * rr = 2 * pi * r, r)      r = rr * x
Define r(x) = rr * x                       Done
Define h(x) = sqrt(rr^2 - (r(x))^2)        Done
Define v(x) = 1/3 * pi * (r(x))^2 * h(x)   Done
...ine v(x) = 1/3 * pi * (r(x))^2 * h(x)
MAIN RAD AUTO FUNC 4/30
  
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
Define r(x) = rr * x                       Done
Define h(x) = sqrt(rr^2 - (r(x))^2)        Done
Define v(x) = 1/3 * pi * (r(x))^2 * h(x)   Done
solve(d/dx(v(x)) = 0, x) | x > 0 and rr > 0
                                           x = sqrt(6)/3
...<d(v(x), x) = 0, x | x > 0 and rr > 0
MAIN RAD AUTO FUNC 5/30
  
```

The maximum for $x = \frac{\sqrt{6}}{3} \approx 0.8165$ corresponds to the angle $\alpha \approx 293.9^\circ$.

Exercise

Imagine that you fold a cone using the sector that was removed from the original circular piece of paper. The total volume of the two cones will be given by $f(x) = V(x) + V(1-x)$. Determine the maximum volume using non-CAS methods.

Solution

It is not easy to find the solution symbolically and voyage 200 fails to do it because it cannot handle cubic equations symbolically. By using a graphic or numerical tool we obtain the answer $x \approx 0.3240$ or $x \approx 0.6760$. It is difficult to see the maxima on the graph because f is "almost constant" when $0.25 < x < 0.75$. There is a relative minimum at $x = 1/2$.

5.7 Ages 17-19 – Elimination of parameters and substitution with computer algebra

5.7.1 Introduction

Starting with the geometrical concept of parametric equations of lines and planes, we illustrate the method of elimination to obtain a cartesian equation. This elimination can be done in a direct and simple way, close to the meaning of the elimination process, by using the procedures "solve" and "substitute" of a Computer Algebra System (the basic algebraic manipulations of formulas).

Without CAS, this method is difficult to realize by hand for a plane. Therefore it was necessary to introduce in advance more elegant (but also more sophisticated) algebraic techniques like determinants. The result was that for a lot of students the meaning of the elimination process disappeared behind these algebraic manipulations.

Later on in the educational process, we have the opportunity to show the equivalence and strength of the new algebraic techniques.

These ideas will be illustrated using a few (geometric) examples.

5.7.2 A line in space

Start at the point $(2,5,1)$ and take t "steps" of vector $(1,4,9)$ to reach a point (x,y,z) . As t runs over \mathbb{R} , we can "see" how a line L of points (x,y,z) is generated.

Parametric equations of L are the algebraic translation of the geometric description of L :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad \text{or} \quad \begin{cases} x = 2 + t \\ y = 5 + 4t \\ z = 1 + 9t \end{cases} .$$

For $t = 0$ we find the point $(2,5,1)$ and for $t = 2$ the point $(4,13,19)$ on the line.

Alternatively, does the point $P(13, 49, 100)$ lie on L ? In other words, can we find a convenient step t in our parametric description to reach the point P ? We know P will lie on L if the system

$$\begin{cases} 13 = 2 + t \\ 49 = 5 + 4t \\ 100 = 1 + 9t \end{cases}$$

has a solution for t . We can verify this by solving the first equation for t (this yields $t = 11$) and then substituting this value into the two last equations to check if the equations are true for the given value of t . These conditions are fulfilled, thus P lies on L .

The same procedure can be followed for an arbitrary point (x,y,z) in space. The point (x,y,z) will lie on the line if the system

$$\begin{cases} x = 2 + t \\ y = 5 + 4t \\ z = 1 + 9t \end{cases}$$

has a solution for t . Solving the first equation for t yields $t = x - 2$.

This solution must validate the second and the third equation:

$$\begin{cases} y = 5 + 4(x - 2) \\ z = 1 + 9(x - 2) \end{cases} \quad \text{or} \quad \begin{cases} y = 4x - 3 \\ z = 9x - 17 \end{cases}$$

We get two conditions for the point (x,y,z) , these conditions are the cartesian equations of L ; the line is described as the intersection of two planes.

Finding cartesian equations of a line means that we have to find the conditions for the coordinates (x,y,z) of a general point in order that the system would have a solution for t . To do this we *eliminate* the parameter t .

CAS can be used to perform the elimination (but calculation by hand is faster) :

```

F1  F2  F3  F4  F5  F6
┌───┬───┬───┬───┬───┬───┐
│ Algebra Calc Other PrgmIO Clean Up │
├───┴───┴───┴───┴───┴───┤
│ x = 2 + t → e1                x = t + 2 │
│ y = 5 + 4 · t → e2            y = 4 · t + 5 │
│ z = 1 + 9 · t → e3            z = 9 · t + 1 │
│ solve(e1 and e2 and e3, t)    │
│   t =  $\frac{z-1}{9}$  and x =  $\frac{z+17}{9}$  and y =  $\frac{4 \cdot z + 41}{9}$  │
│ solve(e1 and e2 and e3, {t y z}) │
│   t = x - 2 and y = 4 · x - 3 and z = 9 · x - 17 │
├───┴───┴───┴───┴───┴───┤
│ ve(e1 and e2 and e3, {t, y, z}) │
├───┴───┴───┴───┴───┴───┤
│ MAIN          RAD AUTO          FUNC 5/30 │
└───┴───┴───┴───┴───┴───┘

```

```

F1  F2  F3  F4  F5  F6
┌───┬───┬───┬───┬───┬───┐
│ Algebra Calc Other PrgmIO Clean Up │
├───┴───┴───┴───┴───┴───┤
│ solve(e1 and e2 and e3, {t x y}) │
│   t =  $\frac{z-1}{9}$  and x =  $\frac{z+17}{9}$  and y =  $\frac{4 \cdot z + 41}{9}$  │
│ solve(e1 and e2 and e3, {t x z}) │
│   t =  $\frac{y-5}{4}$  and x =  $\frac{y+3}{4}$  and z =  $\frac{9 \cdot y - 41}{4}$  │
│ solve(e1 and e2 and e3, {t z}) │
│   t = x - 2 and y = 4 · x - 3 and z = 9 · x - 17 │
├───┴───┴───┴───┴───┴───┤
│ MAIN          RAD AUTO          FUNC 3/30 │
└───┴───┴───┴───┴───┴───┘

```

Solving for t (given x, y, z) with the instruction `solve(e1 and e2 and e3, t)` yields the extra conditions $x = \frac{z+17}{9}$ and $y = \frac{4z+41}{9}$. This illustrates the concept of elimination.

Solving for t, y and z (given x) with `solve(e1 and e2 and e3, {t, y, z})` yields the same planes $y = 4x - 3$ and $z = 9x - 17$ which were obtained in our calculation by hand, but we lose the underlying idea of elimination!

5.7.3 Cartesian equation of a plane

Starting at a given point, we now have to take steps in two different directions to reach every point of a plane.

Start at the point $(6,9,1)$ and take r “steps” of vector $(-4,12,5)$ followed by s steps of vector $(1,3,5)$ to reach a point (x,y,z) . As r and s run over \mathbb{R} , we can “see” how a plane α of points (x,y,z) is generated. For integer values of r and s , we reach the grid points of a grid on α defined by the given vectors.

Parametric equations of α are the algebraic translation of the geometric description of α :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} + r \begin{bmatrix} -4 \\ 12 \\ 5 \end{bmatrix} + s \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{cases} x = 6 - 4r + s \\ y = 9 + 12r + 3s \\ z = 1 + 5r + 5s \end{cases}$$

Following the same procedure as for a line, students should be able to investigate whether a given point lies on the plane or not.

Generalizing that procedure for an arbitrary point (x,y,z) in space, the point (x,y,z) will lie on the plane if the system

$$\begin{cases} x = 6 - 4r + s \\ y = 9 + 12r + 3s \\ z = 1 + 5r + 5s \end{cases}$$

has a solution for r and s .

We now use the two first equations, solve for r and s and then substitute the resultant expressions into the third equation. Where possible get rid of the denominators.

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
■ x = 6 - 4·r + s → e1          x = -4·r + s + 6
■ y = 9 + 12·r + 3·s → e2      y = 12·r + 3·s + 9
■ z = 1 + 5·r + 5·s → e3      z = 5·r + 5·s + 1
■ solve(e1 and e2, {r, s})
  r =  $-\frac{(3 \cdot x - y - 9)}{24}$  and s =  $\frac{3 \cdot x + y - 27}{6}$ 
solve(e1 and e2, {r, s})
MAIN          RAD AUTO          FUNC 4/30
  
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
■ e3 | r =  $-\frac{(3 \cdot x - y - 9)}{24}$  and s =  $\frac{3 \cdot x + y - 27}{6}$ 
  z =  $\frac{15 \cdot x}{8} + \frac{25 \cdot y}{24} - \frac{157}{8}$ 
■  $\left( z = \frac{15 \cdot x}{8} + \frac{25 \cdot y}{24} - \frac{157}{8} \right) \cdot 24$ 
  24·z = 45·x + 25·y - 471
ans(1)*24
MAIN          RAD AUTO          FUNC 2/30
  
```

The plane α has the equation $45x + 25y - 24z - 471 = 0$.

This procedure is very close to the meaning of the elimination process.

The elimination can also be done in one CAS-step by solving the three equations for r and s (given x, y, z) using the instruction `solve(e1 and e2 and e3, {r, s})`. The last equation gives the condition on (x, y, z) for the existence of a solution for r and s .

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
■ solve(e1 and e2 and e3, {r, s})
   $\left( \frac{4 \cdot z + 11}{5} \right)$  and x =  $-\frac{(25 \cdot y - 3 \cdot (8 \cdot z + 157))}{45}$ 
solve(e1 and e2 and e3, {r, s})
MAIN          RAD AUTO          FUNC 1/30
  
```

Solving for r, s and z (given x and y) with `solve(e1 and e2 and e3, {r, s, z})` yields the same plane $45x + 25y - 24z - 471 = 0$ but has a different algebraic meaning, losing the idea of elimination.

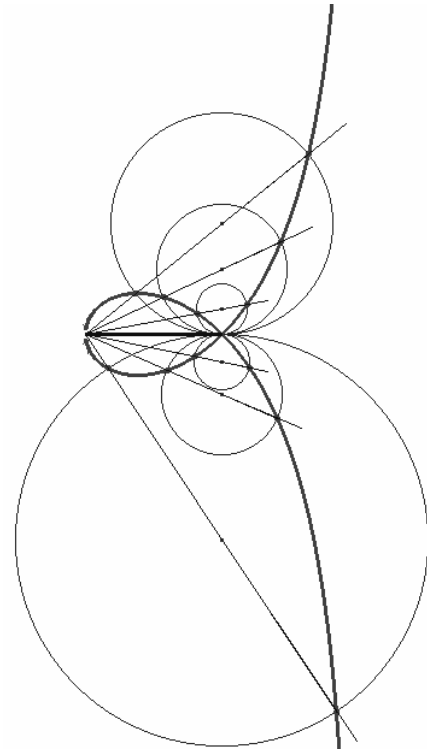
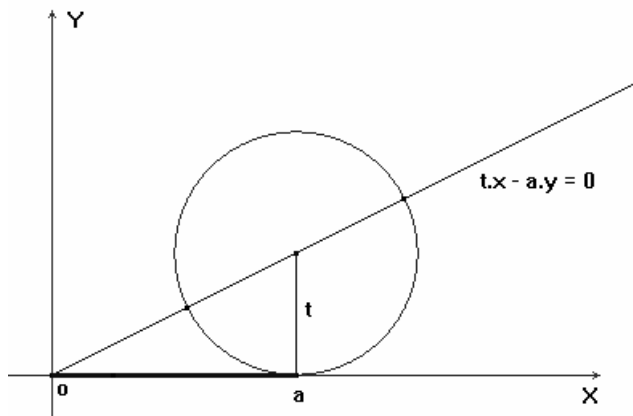
```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
■ solve(e1 and e2 and e3, {r, s, z})
   $s = \frac{3 \cdot x + y - 27}{6}$  and z =  $\frac{45 \cdot x + 25 \cdot y - 471}{24}$ 
solve(e1 and e2 and e3, {r, s, z})
MAIN          RAD AUTO          FUNC 1/30
  
```

5.7.4 The strophoid

Given a line segment with length a , consider a circle tangent to the line segment at one of its endpoints. Draw a line through the other endpoint and the centre of the circle. The locus of the points of intersection of this line with the circle, by allowing the circle to vary, is called a strophoid.

Here we follow the same idea of elimination in the context of plane analytic geometry and the geometric description of loci. Parametric equations are the algebraic translation of the geometric construction process and from these equations we find the cartesian equation by elimination of the parameter(s).

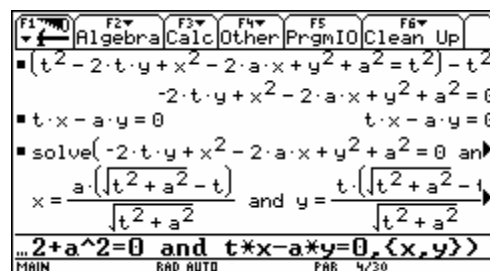
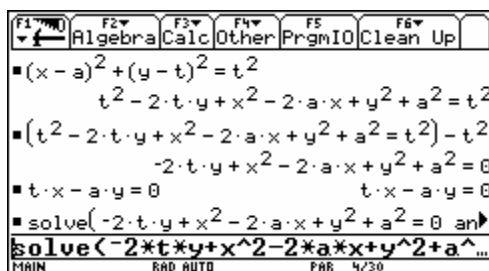


The locus consists of the points of intersection of the associated curves: the line $t \cdot x - a \cdot y = 0$ and the circle $(x - a)^2 + (y - t)^2 = t^2$, when choosing the subjoined coordinate system.

The classical procedure consists of eliminating the parameter t directly from the system of equations of the associated curves (see next page), because it is too difficult to find the (algebraically equivalent) parametric equations by hand.

The difficulty of calculation can be overcome when calculations are left to a calculator with CAS. It is more natural to look for the points of intersection of the associated curves which gives the parametric equations of the curve.

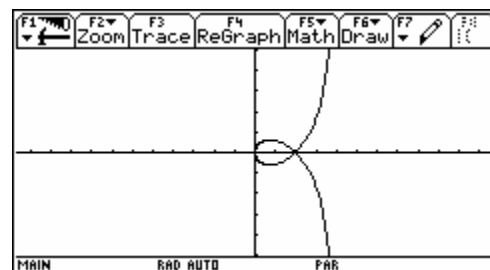
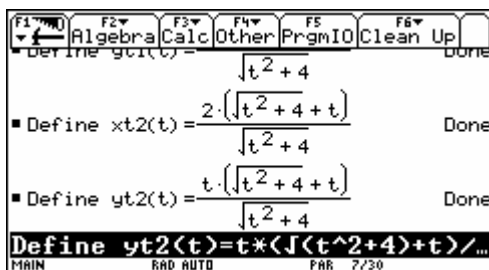
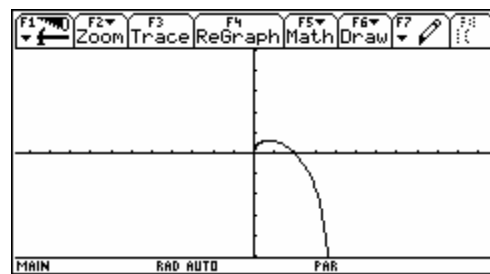
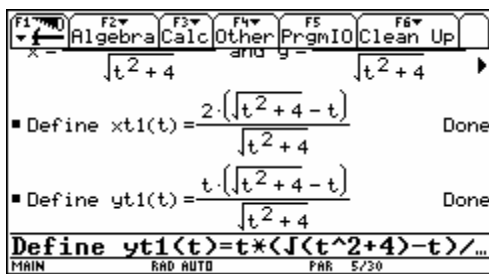
We enter the equations of the circle and the line and solve for x and y :



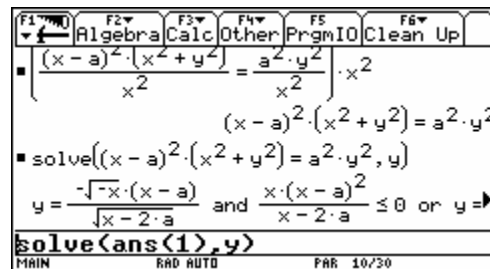
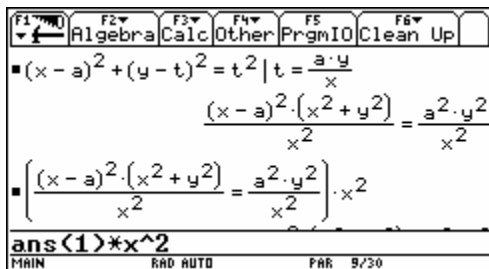
Since the circle and the line have two points of intersection, we find two solutions (connected with "or"):

$$x = \frac{a(\sqrt{t^2 + a^2} - t)}{\sqrt{t^2 + a^2}} \text{ and } y = \frac{t(\sqrt{t^2 + a^2} - t)}{\sqrt{t^2 + a^2}} \text{ or } x = \frac{a(\sqrt{t^2 + a^2} + t)}{\sqrt{t^2 + a^2}} \text{ and } y = \frac{t(\sqrt{t^2 + a^2} + t)}{\sqrt{t^2 + a^2}}$$

Now draw the curve with these parametric equations for $a = 2$:

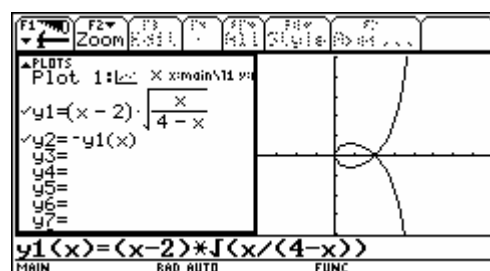
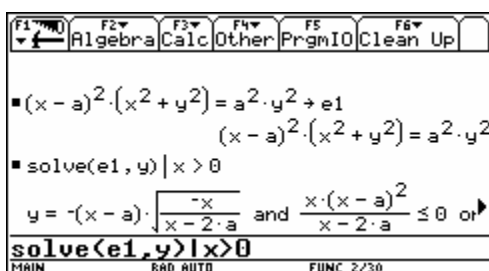


By eliminating the parameter t from the system of the associated curves, we expect the cartesian equation of the strophoid.

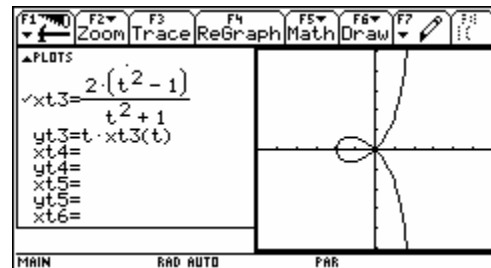
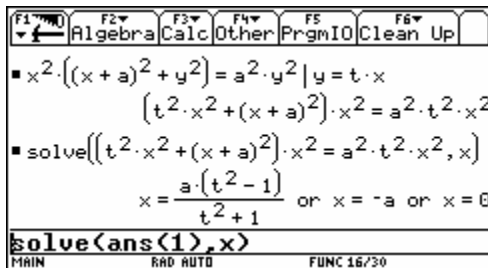


However an incorrect result is obtained, namely the functions $y = \pm \frac{(x-a)\sqrt{-x}}{\sqrt{x-2a}}$, with the null set as

domain! By adding the condition $x > 0$ we get the correct functions $y = \pm(x-a)\sqrt{\frac{-x}{x-2a}}$, yielding the same curve for $a = 2$ as the parametric representation:



Now we replace x by $x + a$ in the cartesian equation $(x - a)^2(x^2 + y^2) = a^2y^2$ in order to translate the double point $(a, 0)$ of the curve to the origin of the coordinate system. Next we intersect the curve with the line $y = t \cdot x$. We obtain one parametric representation for the whole strophoid by calculating the coordinates (x, y) of the point of intersection as a function of the parameter t . Check this with a graph for $a = 2$:



5.7.5 Conclusions

Thanks to computer algebra systems it has become possible to eliminate parameters directly, without the use of determinants. This enables the treatment of elimination at an earlier stage and there is the additional advantage that the meaning of the concept elimination is clarified.

The student can quickly draw a graph and see that the parametric and cartesian representations yield the same curve. The choice of the window and the parameter interval is very important to obtain a nice graph. One needs to take the necessary time to discuss this with students as this provides insight into the geometrical meaning of a parameter.

One can "only" manipulate formulas with "computer algebra" and a computer cannot "see" the meaning of a letter in a given equation. It cannot recognise whether a letter represents a constant, a variable or a parameter. A CAS cannot determine what is given or what is asked, or what the set of numbers we are working with is. These are essential aspects of algebra (fortunately?) reserved for the human brain.

CAS stimulates mathematical reasoning, forces us to reflect on manipulation of formulas and to state explicitly the conditions that are often implicitly assumed when we work with paper and pencil.

5.8 Ages 17-19 – CAS your girl friend

A young man is saying good-bye to his girl friend at the platform just outside the last door at the back of a train. When the train signals departure he is kissing her so warmly that he does not hear the signal. We assume that the train has a constant acceleration of $a = 0.40 \text{ m/s}^2$ and that he starts running at a constant speed 6.0 s after the train began accelerating. Can he catch the train?

All distances are in meters and time in seconds.

The train starts at $t = 0$ at the position $y = 0$.

The young man runs at the constant speed $v \text{ m/s}$.

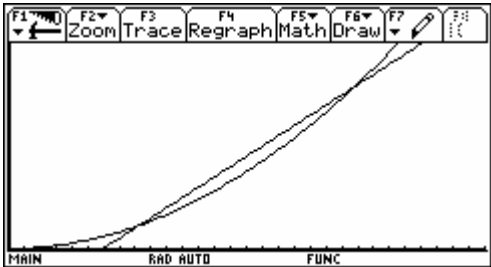
The position of the train door at time t is given by: $y = y1 = \frac{1}{2}at^2 = 0.20 \cdot t^2$.

The position of the young man at time t is given by: $y = y2 = v \cdot (t - 6)$.

We will now analyse this situation in three ways.

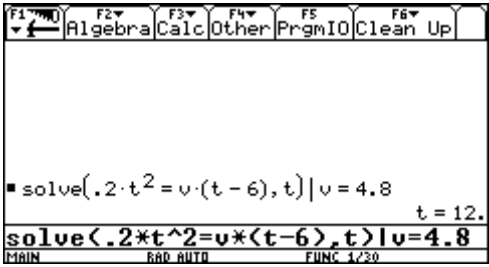
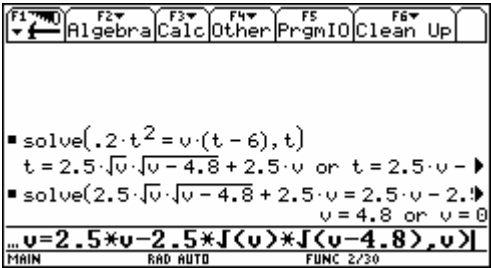
5.8.1 CAS the train 1 (using no calculus)

The graphs below show the distances as a function of time if the man runs at $v = 6.0 \text{ m/s}$:



We see that in this case he has 2 opportunities to catch the train. If he runs too slowly, he will not be able to catch the train. The minimum speed v_{\min} can be found by using the fact that in this case the equation $y1 = y2$ has one and only one solution for t . First solve the equation $y1 = y2$ with any speed v to find the two solutions for t . Then find v such that the two t -values are equal.

Thus the minimum speed in order to catch the train is $v_{\min} = 4.8 \text{ m/s}$ (the solution $v = 0$ makes no sense). Again, we solve the equation $y1 = y2$ for t , this time with $v = 4.8$.



We can now conclude that the minimum running speed required for the man to catch the train is 4.8 m/s. At that speed he will catch the train 12 seconds after the train started, after he has been running for 6 seconds and covered a distance of 28.8 m. We can show this in two ways:

He runs for 6 seconds: $4.8 \text{ m/s} \cdot 6 \text{ s} = 28.8 \text{ m}$

The train travels for 12 seconds: $0.2 \text{ m/s}^2 \cdot (12 \text{ s})^2 = 28.8 \text{ m}$.

If the man runs faster than 4.8 m/s, he will have two opportunities to catch the train.

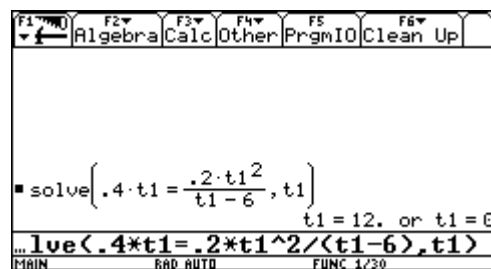
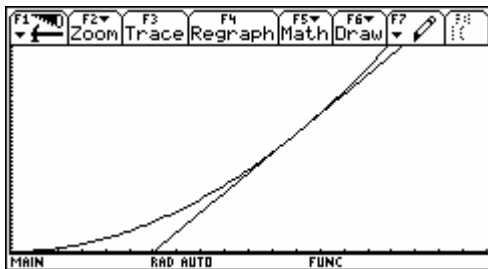
5.8.2 CAS the train 2 (using calculus)

The young man is running at $v = v_{\min}$ and the train has the same speed when he catches it.

Position of train door at time t : $y = \frac{1}{2}at^2 = 0.20 \cdot t^2$.

Speed of train at time t : $v = y'(t) = 0.40 \cdot t$.

The speed of the young man can be written as: $v = \frac{0.20 \cdot t_1^2}{t_1 - 6}$, where t_1 is the time when he catches the train.

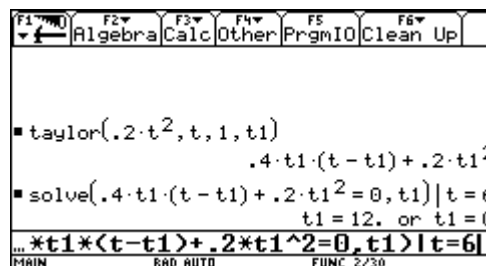


Once again we find that he catches the train $t_1 = 12$ seconds after it started.

5.8.3 CAS the train 3 (using calculus)

Below we find the equation of the tangent line at the time t_1 as the Taylor polynomial of order 1.

Then solve for t_1 requiring that $y = 0$ when $t = 6$:



Once again we find that he catches the train $t_1 = 12$ seconds after it started.

Remark: The model is not fully realistic. You might for instance introduce an acceleration period for the young man before he reaches the constant speed.

5.9 Ages 17-19 – A special attribute of triangles with their vertices on an equilateral hyperbola (C4)

This contribution is part of a paper submitted by M.Gouy, G. Huvent and A.Ladureau “Autour des triangles inscrits sur une hyperbole équilatère”. It is a fine example of a meaningful use of CAS in secondary school. All basics are well known from analytic geometry working with or without vectors. The special problems can be solved very easily, even by-hand, but finding a conjecture would require some boring calculations and proving the conjecture would involve large expressions when working with linear equations. The authors present their solution on various platforms (TI, DERIVE and Cabri). Following is the TI-version with some additional comments.

- a) Take any three points A, B, C on an equilateral hyperbola and find the orthocenter O of the triangle $\triangle ABC$. Do you notice any interesting results?

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
1/x → h(x) Done
[- 1/2 h(-1/2)] → a [- 1/2 -2]
[2 h(2)] → b [2 1/2]
[4 h(4)] → c [4 1/4]
[x y] → c [x y]
dotP(o - a, c - b) = 0 → ha
solve(ha and hb, {x, y})
MAIN RAD AUTO FUNC 4/8

```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
dotP(o - a, c - b) = 0 → ha
2 · x - y/4 + 1/2 = 0
dotP(o - b, c - a) = 0 → hb
9 · x/2 + 9 · y/4 - 81/8 = 0
solve(ha and hb, {x, y})
x = 1/4 and y = 4
solve(ha and hb, {x, y})
MAIN RAD AUTO FUNC 8/30

```

We can see that the orthocenter is a point on the hyperbola. Is this always the case?

Instead of considering some other triangles we will perform another form of generalization by taking randomly chosen points on the hyperbola and a random equilateral hyperbola. We then find a formula for the co-ordinates of the orthocenter. It could be that this formula is well known by the students from earlier investigations on the triangle.

On this occasion the students are encouraged to work with self defined functions and to demonstrate their competence in using the tool by choosing appropriate variable names (eg. $y1$ cannot be used because it is a reserved system variable).

- b) Take any randomly chosen points with $(-8 \leq x \leq 8)$ on a random equilateral hyperbola $y(x) = \frac{k}{x}$ with k a random integer $-6 \leq k \leq 6$ and test this feature once more. First find a formula for the co-ordinates of the orthocenter of a triangle.

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
[x1 yy1] → a : [x2 yy2] → b : [x3 yy3] → c
dotP(o - a, c - b) = 0 → ha
-(x2 - x3) · x - (yy2 - yy3) · y + x1 · (x2 - x3) →
dotP(o - b, c - a) = 0 → hb
-(x1 - x3) · x - (yy1 - yy3) · y + x1 · x2 - x2 · x3 →
solve(ha and hb, {x, y})
-(x1 · (x2 · (yy1 - yy2) - x3 · (yy1 - yy3)) +
MAIN RAD AUTO PAR 14/30

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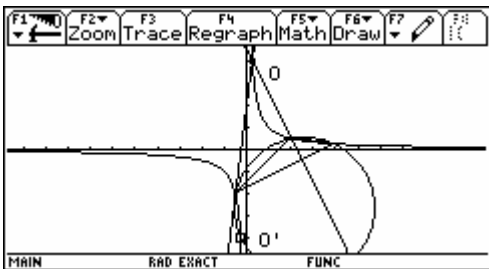
F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
solve(ha and hb, {x, y})
x = - (x1 · (x2 · (yy1 - yy2) - x3 · (yy1 - yy3)) +
- yy2 · yy3) → oc(x1, yy1, x2, yy2, x3, yy3)
Done
oc(-1/2, h(-1/2), 2, h(2), 4, h(4))
(-1/2, h(-1/2), 2, h(2), 4, h(4))
MAIN RAD AUTO PAR 14/30

```


The students will find out that the hyperbola surprisingly passes through Point O. Instead of recalculation with another triangle they can immediately consider the general proof.

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\frac{1}{x} \rightarrow h(x)$ Done $[2 \ h(2)] \rightarrow a : [-1/2 \ h(-1/2)] \rightarrow b : [4 \ 1/4]$ $[x \ y] \rightarrow cc$ [x y] $\text{dotP}(cc - \frac{b+c}{2}, b-c) = 0 \rightarrow pba$ $\frac{-9 \cdot x}{2} - \frac{9 \cdot y}{4} + \frac{189}{32} = 0$					
MAIN	RAD EXACT	FUNC 24/24			

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\text{dotP}(cc - \frac{a+c}{2}, a-c) = 0 \rightarrow pbb$ $-2 \cdot x + \frac{y}{4} + \frac{189}{32} = 0$ $\text{solve}(pba \text{ and } pbb, \{x \ y\})$ $x = 21/8 \text{ and } y = -21/8$ $\text{norm}([21/8 \ -21/8] - a)$ $\frac{5 \cdot \sqrt{26}}{8}$					
MAIN	RAD EXACT	FUNC 24/30			



F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\text{Circle } 21/8, -21/8, \frac{5 \cdot \sqrt{26}}{8}$ Done $(x - 21/8)^2 + (y + 21/8)^2 = (\frac{5 \cdot \sqrt{26}}{8})^2 \rightarrow \text{ccirc}$ $x^2 - \frac{21 \cdot x}{4} + y^2 + \frac{21 \cdot y}{4} + \frac{441}{32} = \frac{325}{32}$ $\text{ccircle} x = -1/4 \text{ and } y = -4$ TRUE					
MAIN	RAD EXACT	FUNC 15/24			

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$[x1 \ \frac{k}{x1}] \rightarrow a : [x2 \ \frac{k}{x2}] \rightarrow b : [x3 \ \frac{k}{x3}] \rightarrow c$ [x3 k/x3] $\text{dotP}(cc - \frac{b+c}{2}, b-c) = 0 \rightarrow pba$ $\text{dotP}(cc - \frac{a+c}{2}, a-c) = 0 \rightarrow pbb$					
MAIN	RAD EXACT	FUNC 24/30			

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\text{solve}(pba \text{ and } pbb, \{x \ y\})$ Error: Break $\text{solve}(pba, x)$ $x = \frac{2 \cdot k \cdot x2 \cdot x3 \cdot y - (k^2 - x2^2 \cdot x3^2) \cdot (x2 + x3)}{2 \cdot x2^2 \cdot x3^2}$ $pbb x = \frac{2 \cdot k \cdot x2 \cdot x3 \cdot y - (k^2 - x2^2 \cdot x3^2) \cdot (x2)}{2 \cdot x2^2 \cdot x3^2}$					
MAIN	RAD EXACT	FUNC 25/30			

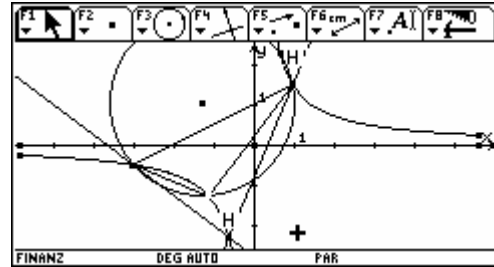
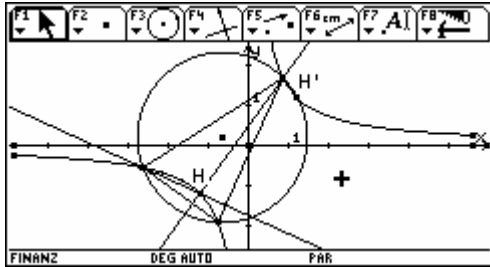
F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\text{solve} \left(\frac{(x1 - x3) \cdot (2 \cdot k \cdot x1 \cdot (x1 - x2) \cdot x2 \cdot x3 \cdot y)}{2 \cdot k \cdot x1 \cdot x2 \cdot x3} \right)$ $y = \frac{k^2 \cdot (x1 \cdot (x2 + x3) + x2 \cdot x3) + x1^2 \cdot x2^2 \cdot x3}{2 \cdot k \cdot x1 \cdot x2 \cdot x3}$ $\frac{(x2 + x3) + x2 \cdot x3 + x1^2 \cdot x2^2 \cdot x3^2}{2 \cdot k \cdot x1 \cdot x2 \cdot x3} \rightarrow ycenter$					
MAIN	RAD EXACT	FUNC 24/30			

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\frac{(x2 + x3) + x2 \cdot x3 + x1^2 \cdot x2^2 \cdot x3^2}{2 \cdot k \cdot x1 \cdot x2 \cdot x3} y = ycenter$ $\frac{k^2 + x1 \cdot (x1 + x2 + x3) \cdot x2 \cdot x3}{2 \cdot x1 \cdot x2 \cdot x3} \rightarrow xcenter$					
MAIN	RAD EXACT	FUNC 24/30			

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$(x - xcenter)^2 + (y - ycenter)^2 = (\text{norm}(a - b))^2$ $4 \cdot k^2 \cdot x1^2 \cdot x2^2 \cdot x3^2 \cdot x^2 - 4 \cdot k^2 \cdot (k^2 + x1 \cdot (x1 + x2 + x3) \cdot x2 \cdot x3) \cdot x + \dots$ $\text{oc}(x1, \frac{k}{x1}, x2, \frac{k}{x2}, x3, \frac{k}{x3})$					
MAIN	RAD EXACT	FUNC 24/30			

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\text{oc}(x1, \frac{k}{x1}, x2, \frac{k}{x2}, x3, \frac{k}{x3})$ $\text{circ}_c x = \frac{k^2}{x1 \cdot x2 \cdot x3} \text{ and } y = \frac{x1 \cdot x2 \cdot x3}{k}$ TRUE					
MAIN	RAD EXACT	FUNC 1/23			

A fine accomplishment of this investigation is the realization that a Dynamic Geometry Tool such as Geometer's Sketchpad or Cabri can be used to represent this problem. Both of these programs can be used on the Voyage 200. Doing this we demonstrate the multiple representations of one problem – analytical, graphical and dynamic– addressing various conceptual levels of the students.



*M.Gouy, G.Huvent & A.Ladureau, Autour des triangles inscrits sur une hyperbole équilatère, 2003
The full article can be found at perso.wanadoo.fr/gery.huvent in the rubric IREM de Lille.*

5.10 Ages 17-19 – Exploring functions with CAS

a) Consider the functions $c_n(x) = \cos(n \cdot \arccos x)$ with $n \in \mathbb{N}_0$:

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	cos(cos ⁴ (x))				x
■	tExpand(cos(2·cos ⁴ (x)))				2·x ² - 1
■	tExpand(cos(3·cos ⁴ (x)))				4·x ³ - 3·x
■	tExpand(cos(4·cos ⁴ (x)))				8·x ⁴ - 8·x ² + 1
■	tExpand(cos(5·cos ⁴ (x)))				16·x ⁵ - 20·x ³ + 5·x
tExpand(cos(5cos⁴(x)))					
MAIN RAD AUTO PAR 5/30					

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	tExpand(cos(6·cos ⁴ (x)))				16·x ⁵ - 20·x ³ + 5·x
■	tExpand(cos(7·cos ⁴ (x)))				32·x ⁶ - 48·x ⁴ + 18·x ² - 1
■	tExpand(cos(8·cos ⁴ (x)))				64·x ⁷ - 112·x ⁵ + 56·x ³ - 7·x
■	tExpand(cos(8·cos ⁴ (x)))				128·x ⁸ - 256·x ⁶ + 160·x ⁴ - 32·x ² + 1
tExpand(cos(8cos⁴(x)))					
MAIN RAD AUTO PAR 8/30					

Conjecture

$c_n(x)$ is a polynomial function of degree n . The function is even if n is even and odd if n is odd.

b) Consider the functions $s_n(x) = \sin(n \cdot \arcsin x)$ with $n \in \mathbb{N}_0$:

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	sin(sin ⁴ (x))				x
■	tExpand(sin(2·sin ⁴ (x)))				2·x·√1-x ²
■	tExpand(sin(3·sin ⁴ (x)))				3·x - 4·x ³
■	tExpand(sin(4·sin ⁴ (x)))				4·x·√1-x ² - 8·x ³ ·√1-x ²
tExpand(sin(4sin⁴(x)))					
MAIN RAD AUTO PAR 4/30					

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	tExpand(sin(5·sin ⁴ (x)))				4·x·√1-x ² - 8·x ³ ·√1-x ²
■	tExpand(sin(6·sin ⁴ (x)))				16·x ⁵ - 20·x ³ + 5·x
■	tExpand(sin(7·sin ⁴ (x)))				32·x ⁵ ·√1-x ² - 32·x ³ ·√1-x ² + 6·x·√1-x ²
■	tExpand(sin(7·sin ⁴ (x)))				-64·x ⁷ + 112·x ⁵ - 56·x ³ + 7·x
tExpand(sin(7sin⁴(x)))					
MAIN RAD AUTO PAR 7/30					

Conjecture

$s_n(x)$ is a polynomial function of degree n if n is odd, $s_n(x) = c_n(x)$ for $n = 1, 5, 9, \dots$ and

$s_n(x) = -c_n(x)$ for $n = 3, 7, 11, \dots$

$s_n(x)$ is $\sqrt{1-x^2}$ times a polynomial of degree $n-1$ if n is even.

The function $s_n(x)$ is odd for all $n \in \mathbb{N}_0$.

c) Consider the functions $t_n(x) = \tan(n \cdot \arctan x)$ with $n \in \mathbb{N}_0$:

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	tan(tan ⁴ (x))				x
■	tExpand(tan(2·tan ⁴ (x)))				$\frac{-2 \cdot x}{x^2 - 1}$
■	tExpand(tan(3·tan ⁴ (x)))				$\frac{x^3 - 3 \cdot x}{3 \cdot x^2 - 1}$
tExpand(tan(3tan⁴(x)))					
MAIN RAD AUTO PAR 3/30					

F1	F2	F3	F4	F5	F6
←	Algebra	Calc	Other	PrgmIO	Clean Up
■	tExpand(tan(4·tan ⁴ (x)))				$\frac{4 \cdot x - 4 \cdot x^3}{x^4 - 6 \cdot x^2 + 1}$
■	tExpand(tan(5·tan ⁴ (x)))				$\frac{x^5 - 10 \cdot x^3 + 5 \cdot x}{5 \cdot x^4 - 10 \cdot x^2 + 1}$
tExpand(tan(5tan⁴(x)))					
MAIN RAD AUTO PAR 2/30					

Conjecture

$t_n(x)$ is an odd rational function, with an odd numerator and even denominator.

If n is even, the denominator has degree n and the numerator has degree $n - 1$. If n is odd, the numerator has degree n and the denominator has degree $n - 1$.

Proof of the conjectures

(i) Are the functions even or odd?

- $c_n(-x) = \cos(n \cdot \arccos(-x)) = \cos(n \cdot (\pi - \arccos x)) = \cos(n \cdot \pi - n \cdot \arccos x)$

If n is even, $c_n(-x) = \cos(-n \cdot \arccos x) = \cos(n \cdot \arccos x) = c_n(x)$.

If n is odd, $c_n(-x) = \cos(\pi - n \cdot \arccos x) = -\cos(n \cdot \arccos x) = -c_n(x)$.

Conclusion: $c_n(x)$ is even if n is even and odd if n is odd.

- $s_n(-x) = \sin(n \cdot \arcsin(-x)) = \sin(-n \cdot \arcsin x) = -s_n(x)$

Conclusion: the functions $s_n(x)$ and $t_n(x)$ (analogous) are odd.

(ii) $s_n(x) = c_n(x)$ for $n = 1, 5, 9, \dots$ and $s_n(x) = -c_n(x)$ for $n = 3, 7, 11, \dots$

$$s_n(x) = \sin(n \cdot \arcsin x) = \sin\left(n \cdot \left(\frac{\pi}{2} - \arccos x\right)\right) = \sin\left(n \cdot \frac{\pi}{2} - n \cdot \arccos(x)\right)$$

If $n = 1 + 4k$, $s_n(x) = \sin\left(\frac{\pi}{2} + k \cdot 2\pi - n \cdot \arccos(x)\right) = \sin\left(\frac{\pi}{2} - n \cdot \arccos(x)\right) = c_n(x)$

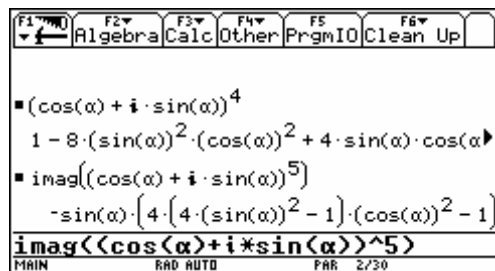
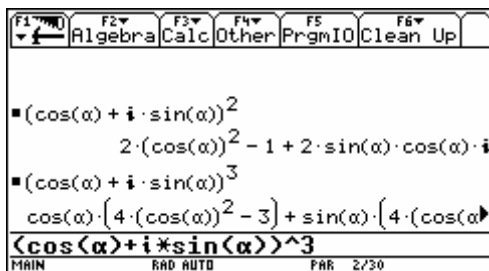
If $n = 3 + 4k$, $s_n(x) = \sin\left(\frac{3\pi}{2} + k \cdot 2\pi - n \cdot \arccos(x)\right) = \sin\left(\frac{3\pi}{2} - n \cdot \arccos(x)\right) = -\sin\left(\frac{\pi}{2} - n \cdot \arccos(x)\right) = -c_n(x)$

(iii) $c_n(x)$ ($n \in \mathbb{N}_0$) and $s_n(x)$ (n odd) are polynomials of degree n .

De Moivre's theorem states that for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$(\cos \alpha + i \cdot \sin \alpha)^n = \cos(n\alpha) + i \cdot \sin(n\alpha).$$

Expanding the left side and equating the real and imaginary parts provides expressions for $\cos(n\alpha)$ and $\sin(n\alpha)$. Then substitute $\alpha = \arccos x$ (or $\alpha = \arcsin x$).

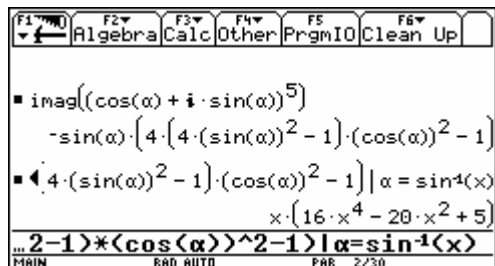


Example:

Substitution of $\alpha = \arcsin x$ in $\sin(5\alpha)$

yields the polynomial

$$s_5(x) = 16x^5 - 20x^3 + 5x$$



The binomial theorem proves the general case:

$$(\cos \alpha + i \cdot \sin \alpha)^n = \sum_{k=0}^n \binom{n}{k} (\cos \alpha)^k \cdot (i \cdot \sin \alpha)^{n-k} = \sum_{k=0}^n \binom{n}{k} (\cos \alpha)^k \cdot (\sin \alpha)^{n-k} \cdot i^{n-k}.$$

Collect the terms with even power $n-k$ of i for the real part $\cos(n \cdot \alpha)$ and substitute

$$\alpha = \arccos x, \text{ then } (\cos \alpha)^k \cdot (\sin \alpha)^{n-k} = x^k \cdot (1-x^2)^{\frac{n-k}{2}}, \text{ a polynomial of degree } n.$$

Collect the terms with odd power $n-k$ of i to find the imaginary part $\sin(n \cdot \alpha)$ and substitute $\alpha = \arcsin x$.

If n is odd, k must be even and $(\cos \alpha)^k \cdot (\sin \alpha)^{n-k} = (1-x^2)^{\frac{k}{2}} \cdot x^{n-k}$ is a polynomial of degree n .

If n is even, k must be odd and $(\cos \alpha)^k \cdot (\sin \alpha)^{n-k} = \sqrt{1-x^2} \cdot (1-x^2)^{\frac{k-1}{2}} \cdot x^{n-k}$.

Therefore $(1-x^2)^{\frac{k-1}{2}} \cdot x^{n-k}$ is a polynomial of degree $n-1$.

(iv) $t_n(x) = \tan(n \cdot \arctan x)$ is a rational function ($n \in \mathbb{N}_0$).

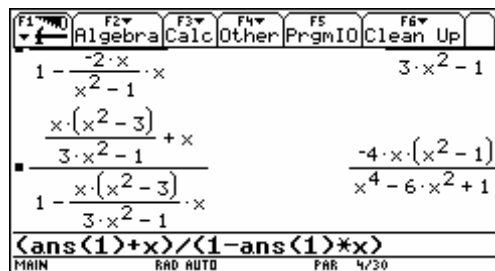
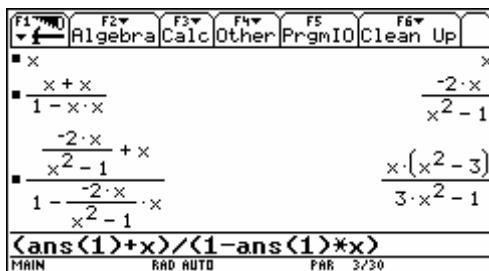
Using the trigonometric formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$, we find that

$$\tan(n\alpha) = \tan((n-1)\alpha + \alpha) = \frac{\tan((n-1)\alpha) + \tan \alpha}{1 - \tan((n-1)\alpha) \cdot \tan \alpha}.$$

The substitution $\alpha = \arctan(x)$ yields the recursion formula

$$t_n(x) = \frac{t_{n-1}(x) + x}{1 - t_{n-1}(x) \cdot x} \quad (n \geq 2 \text{ and } t_1(x) = x).$$

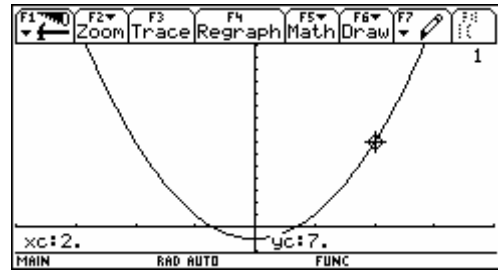
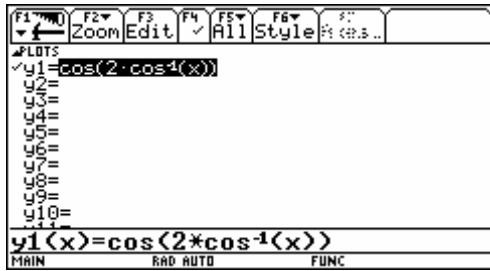
CAS can be used to produce the sequence $t_1(x), t_2(x), t_3(x), \dots$



d) Remarks

- (i) Observe how CAS and formal mathematics can cooperate.
- (ii) The real function $c_2(x) = \cos(2 \cdot \arccos x) = 2x^2 - 1$ has domain $[-1, 1]$.

For CAS, the function has domain \mathbb{R} : working with complex functions, the result of $\arccos(2)$ is non-real, but $\cos(2\arccos(2)) = 7$ is real!



(iii) Knowing that $c_n(x)$ ($n \in \mathbb{N}_0$) and $s_n(x)$ (n odd) are polynomials of degree n , it is possible to find these polynomials with their zeros.

To find the zeros of $c_4(x) = \cos(4 \cdot \arccos x)$,

observe that $0 \leq 4\arccos x \leq 4\pi$ and

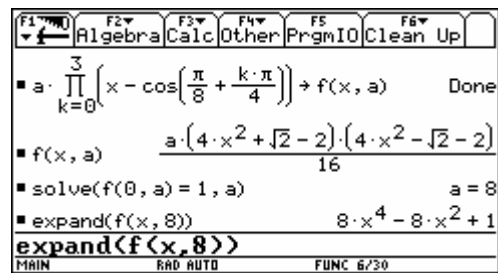
$$\text{if } 4 \cdot \arccos x = \frac{\pi}{2} + k \cdot \pi \quad (k = 0, 1, 2, 3)$$

$$\text{or } \arccos x = \frac{\pi}{8} + k \cdot \frac{\pi}{4} \quad (k = 0, 1, 2, 3).$$

The zeros of $c_4(x)$ are $x_k = \cos\left(\frac{\pi}{8} + k \cdot \frac{\pi}{4}\right)$ ($k = 0, 1, 2, 3$), consequently

$$c_4(x) = a(x - x_0)(x - x_1)(x - x_2)(x - x_3).$$

Find a with $c_4(0) = 1$. Result: $c_4(x) = 8x^4 - 8x^2 + 1$.



5.11 Ages 17-19 – You must still think yourself, CAS can only help

a) A classroom activity and instructions for teachers

- (i) Ask the students to draw the graph of a cubic, e.g. $f(x) = x^3 - 5 \cdot x^2 + 2 \cdot x + 4$. They must all have the same graph.
- (ii) Each student now chooses a linear function $g(x) = p \cdot x + q$, such that the line intersects the cubic three times. Have each student choose a different linear function.
- (iii) Determine the solutions x_1, x_2 , and x_3 to the equation $f(x) = g(x)$ and calculate the sum of the solutions, $x_1 + x_2 + x_3$. The values of the three solutions can be found using either the graph or solve.

The most direct way to obtain the sum is to write: $\text{sum}(\text{zeros}(f(x) - g(x), x))$.

It will definitely surprise the students that even though they have different values for the solutions x_1, x_2 , and x_3 , they will all obtain the same $\text{sum}(x_1 + x_2 + x_3 = 5)$.

- (iv) Repeat the above procedure with another cubic.
- (v) Conjecture
For any cubic $f(x) = x^3 + a \cdot x^2 + b \cdot x + c$ the graph of which intersects a line at three points, the sum of the x -values of the intersection points is $-a$ regardless of which line is drawn.

b) Solution for (v)

Proving the conjecture cannot be done directly with a Voyage 200. If you try to solve the equation $x^3 + a \cdot x^2 + b \cdot x + c = p \cdot x + q$ (*), you won't get anywhere.

You simply do not get an expression for the solutions. The proof is easy – also by hand – but you must have an approach for solving the problem.

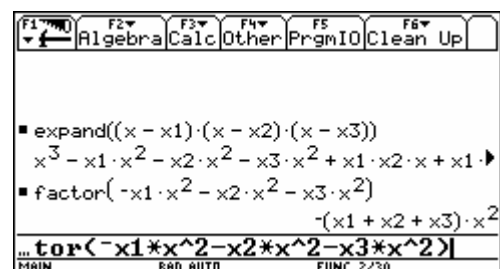
Proof

The solutions x_1, x_2 and x_3 to equation (*) are the zeros of the polynomial $h(x)$ where:

$h(x) = f(x) - g(x) = x^3 + a \cdot x^2 + (b - p) \cdot x + c - q$. This means that

$h(x) = (x - x_1) \cdot (x - x_2) \cdot (x - x_3)$, and at this point many students will choose to use their Voyage 200.

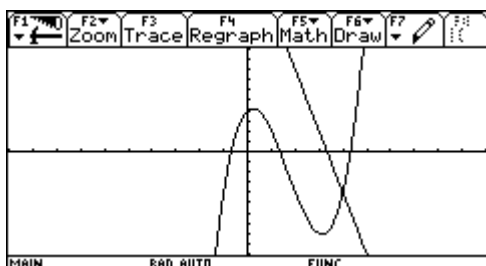
This concludes the proof. The basic reason why we get the same sum for all lines is that the values of p and q do not affect the coefficient of the term of degree 2.



This simple example demonstrates the importance of good mathematical understanding for solving problems in a CAS classroom. Students will not be able to use CAS to solve this problem unless they have good mathematical understanding of the problem.

c) (i) Example

With $f(x) = x^3 - 5 \cdot x^2 + 2 \cdot x + 4$ and $g(x) = -6 \cdot x + 20$ there is only one point of intersection. The equation $f(x) = g(x)$ has one real solution and two imaginary solutions. The proof in b) can be extended to complex numbers without any changes.



```

F1 (Zoom) F2 (Trace) F3 (Regraph) F4 (Math) F5 (Draw) F6 (Pencil) F7 (Eraser)
Algebra Calc Other PrgmIO Clean Up
x^3 - 5 · x^2 + 2 · x + 4 → f(x) Done
20 - 6 · x → g(x) Done
cSolve(f(x) = g(x), x)
x = 1/2 + √15/2 · i or x = 1/2 - √15/2 · i or x = 4
sum(cZeros(f(x) - g(x), x))
sum(cZeros(f(x) - g(x), x))
MAIN RAD AUTO FUNC 4/30

```

(ii) Exercise

With $f(x) = x^3 - 6 \cdot x^2 + 5 \cdot x + 4$ and $g(x) = -7 \cdot x + 12$ we get

```

F1 (Zoom) F2 (Trace) F3 (Regraph) F4 (Math) F5 (Draw) F6 (Pencil) F7 (Eraser)
Algebra Calc Other PrgmIO Clean Up
x^3 - 6 · x^2 + 5 · x + 4 → f(x) Done
-7 · x + 12 → g(x) Done
sum(cZeros(f(x) - g(x), x)) 2
sum(cZeros(f(x) - g(x), x))
MAIN RAD AUTO FUNC 3/30

```

What is wrong? Why is the sum of zeros not equal to 6 ?

(iii) Exercise

Study $f(x) = x^3 - 6 \cdot x^2 + 5 \cdot x + 4$ and $g(x) = -4 \cdot x + 8$ the same way as above, and try to formulate a general theorem about the sum of solutions for any equation $x^3 + a \cdot x^2 + b \cdot x + c = p \cdot x + q$.

d) **Exercise**

Draw the graph of the quartic $f(x) = x^4 - 2 \cdot x^3 - 3 \cdot x^2 + 5 \cdot x - 3$ and choose a linear function $g(x) = p \cdot x + q$ to give four intersection points. Answer questions (1)-(3) given above for this quartic.

6. Final assessment models

6.1 Danish Baccalaureate May 2000 - Mathematics, Written Exam

6.1.1 A-level - 2 hours - Total: 50 points

Paper and pencil only

Question 1 Expand $(2x + y)^2 - 4x(x + y)$.

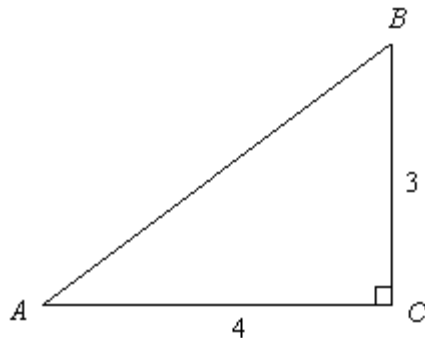
(2 points)

Question 2 Solve the inequality $x^2 + 3x + 2 < 0$.

(4 points)

Question 3 In the triangle ABC , $\angle C = 90^\circ$. The sides are 3 and 4 (see fig.)

(4 points)



Calculate $\sin(A) + \cos(A) + \tan(A)$.

Question 4 Calculate the integral $\int_0^1 3x^5 dx$.

(3 points)

Question 5 A sphere is given by the equation $x^2 + y^2 + z^2 - 2x + 4y - 4z = 16$.

(6 points)

Find the radius of the sphere and the coordinates of the centre.

Given a point $P(4, 2, 2)$ on the sphere, find the equation of the plane tangential to the sphere at P .

Question 6 A random variable X is normally distributed with mean 15 and standard deviation 2.

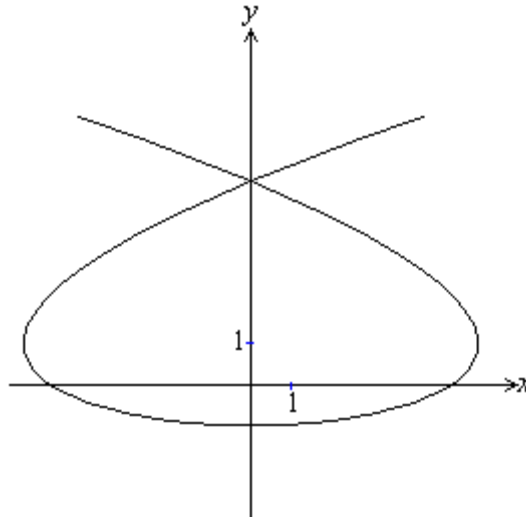
(4 points)

Find the probability $P(14 \leq X \leq 18)$.

Note: Normal distribution paper permitted

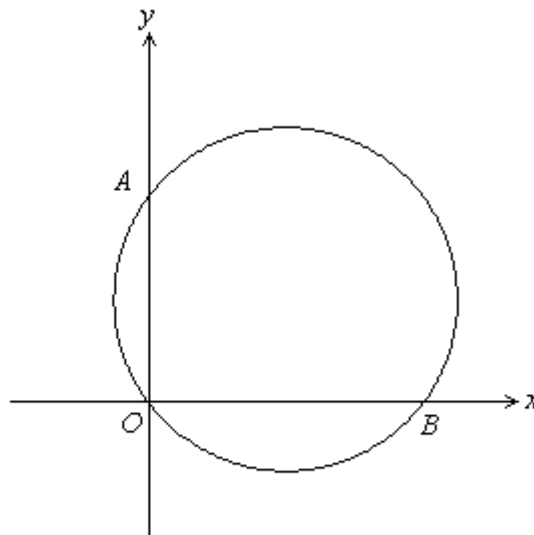
Question 7 The movement of a point $P(x, y)$ as a function of time t is given
(6 points)

by the parametric equations
$$\begin{cases} x = t^3 - 6t \\ y = t^2 - 1 \end{cases} \quad t \in \mathbb{R}.$$



Calculate the coordinates of the intersection points with the y -axis.
Calculate the coordinates of the points on the path where the velocity vector is parallel to one of the axes.

Question 8 The figure below shows a circle with centre in the first quadrant.
(5 points)



The circle passes through the origin $O(0, 0)$ and intersects the axes at $A(0, 3)$ and $B(4, 0)$. Find the equation of the tangent line at B .

Question 9 The size of a population is given by the function $N(t)$, where $N(t)$ is the number of individuals at time t hours. Given that $N(t)$ is a quadratic function with
(5 points) $N'(t) = 4t + 3$ and $N(0) = 800$.

Find the size of the population at the time $t = 10$.

Question 10 Calculate the limits:
(4 points)

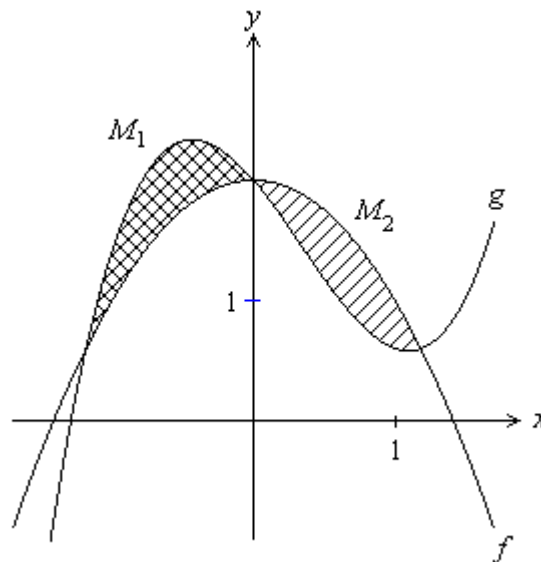
$$\lim_{x \rightarrow 0} \frac{1-x^2}{4-2x} \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{(2+3x)^2-4}{6x}.$$

Question 11 The functions f and g are given by
(7 points)

$$f(x) = -x^2 + 2 \quad \text{and} \quad g(x) = x^3 - x^2 - bx + 2, \quad \text{where } b \text{ is positive.}$$

The graphs of f and g define two regions M_1 and M_2 (see fig.).

Show that M_1 and M_2 have the same area.



6.1.2 A-level - 4 hours - total: 100 points

Books, tables of formulae, graphic calculators, corrected homework etc. permitted

Questions 7a and 7b are optional (pupils must choose one)

CAS version of exam questions no. 3, 4, 6, and 7a . Pupils taking the CAS version may also use TI-89/92, Computer with MathCad etc.

Question 1 (10% , no special CAS version)

In an orthonormal coordinate system, two vectors are given by $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

Calculate t , when $\det(\vec{a}, \vec{b}) = -10$.

For this value of t , calculate the angle between \vec{a} and \vec{b} .

Question 2 (15% , no special CAS version)

In triangle ABC , $\angle A = 20^\circ$, $\angle B = 30^\circ$ and $|AC| = 8$.

Calculate the length of the sides AB and BC .

Calculate the length of the altitude from C .

Calculate the length of the median line from C .

Question 3 (15%) [CAS version in brackets]

In an orthonormal coordinate system, a line l is given by the parametric equations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Find an equation for the plane which passes through the line l and the point $Q(4, -2, 5)$.

A sphere K is given by the equation

$$(x-4)^2 + (y+2)^2 + (z-5)^2 = 11. \quad [(x-a)^2 + (y+2)^2 + (z-5)^2 = 11, \text{ where } a > 2]$$

The sphere K and the line l have exactly one common point P .

Find the coordinates of the point P . [Find a and the coordinates of the point P]

Find an equation of the plane which is tangential to the sphere at the point P .

Question 4 (25%) [CAS version in brackets]

Consider the real function f defined by

$$f(x) = x - 2 \cdot \sqrt{x}, \quad x \geq 0.$$

Determine the zeros of f and the intervals where f is increasing/decreasing.

Draw the graph of f , and determine the range. A region M in the fourth quadrant is bounded by the graph of the function f and the x -axis.

Calculate, using anti-derivatives, the area of M .

Use integration to calculate the volume of the solid of revolution generated by the rotation of M around the x -axis.

[Consider a family of real functions

$$f_a(x) = x - a \cdot \sqrt{x}, \quad x \geq 0,$$

where the parameter a is a positive number.

Determine the zeros of f_2 and the intervals where f_2 is decreasing / increasing.

Draw the graph of f_2 and determine the range. [A region M_a in the fourth quadrant is bounded by the graph of f_a and the x -axis.

Given that the area of M_a is A_a , and the volume of the solid of revolution generated by the rotation of M_a around the x -axis is V_a .

Calculate A_a and V_a .

Calculate a , such that $V_a = \pi \cdot A_a$.]

Question 5 (10 % , no special CAS version)

A manufacturer produces components for computers. The components are produced by the machines A , B , and C , making 60%, 25%, and 15%, respectively, of the total production. Experience shows that 20% of the components produced by A , 15% of the components produced by B , and 5% of the components produced by C , are defective.

Calculate the probability that a component, chosen at random, is defective. A component, chosen at random, appears to be defective. Calculate the probability that it was produced by machine C .

Question 6 (15%) [Extra CAS question in brackets]

Given that a function f , is a solution of the differential equation $\frac{dy}{dx} = \frac{y}{\ln(y)}(x+2)$, $y > 1$,

and that the graph of f passes through the point $P(2,e)$.

Determine an equation of the line which is a tangent to the graph of f at the point P .

Determine the solution f , including the domain of f .

[Extra CAS: Show that the equation $f(x) = f'(x)$ has no solutions.]

Choose question 7a or 7b

Question 7a (10%) [CAS version in brackets]

Consider the function

$$f(x) = \frac{x^3 + 4x^2 - 4x - 17}{x^2 + x - 12} \quad [f(x) = \frac{x^2 + 3x - 4}{x + 1} + \frac{2x^3 - 4x^2 + 7x - 12}{x^2 - 4}.]$$

Find the domain of f .

Find the equation of the asymptotes of f .

Question 7b (10% , no special CAS version)

For the Galapagos Islands, a good approximation for the number of species of terrestrial plants on any island can be determined as a function $N(x) = b \cdot x^a$ of the area x , in square miles.

Given that $N(15) = 68$ and $N(174) = 149$, calculate a and b .

Consider two islands, where the area of one island is 2.5 times the area of the other island.

Calculate the ratio of the number of terrestrial species on these two islands.

6.1.3 Solution (with TI89/92Plus/Voyage 200) of Danish Baccalaureate May 2000 – Mathematics, Written Exam A-level

Format of the solution

Left column: Text of the question, and text of the solution, not a model for pupils, but a short version.

Right column: Category of question and calculator screens.

The left and right columns are to be read together.

Question 1 (10% , no special CAS version)

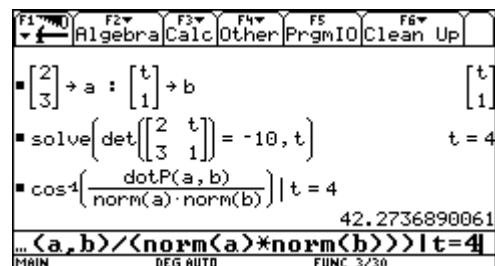
In an orthonormal coordinate system, two vectors are

given by $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

Calculate t , when $\det(\vec{a}, \vec{b}) = -10$.

For this value of t , calculate the angle between \vec{a} and \vec{b} .

Category C0 (CAS-neutral)



Answer to question 1

We store the two given vectors.

The determinant is -10, when $t = 4$.

The angle between the two vectors (42.27°) is found

using the formula $\cos(\angle(\vec{a}, \vec{b})) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$.

Question 2 (15% , no special CAS version)

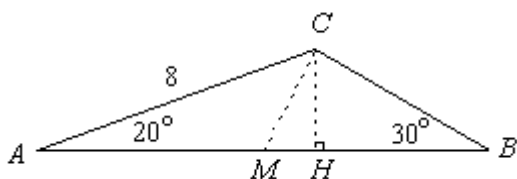
In triangle ABC , $\angle A = 20^\circ$, $\angle B = 30^\circ$ and $|AC| = 8$.

Calculate the length of the sides AB and BC .

Calculate the length of the altitude from C .

Calculate the length of the median line from C .

Answer to question 2



First we store the given sides and angles. We denote the sides by a , b , and c , and the angles by va , vb , and vc . Using the sine relation we find that $AB \approx 12.26$ and $AC \approx 5.47$.

The value for AB is copied and stored in c .

The altitude $CH \approx 2.74$ is found from triangle ACH .

The median segment $CM \approx 3.07$ is found from triangle ACM .

Question 3 (15%) [CAS version]

In an orthonormal coordinate system, a line l is given

by the equations: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$. Find an equation

for the plane α through l and $Q(4, -2, 5)$.

A sphere K is given by the equation

$(x - a)^2 + (y + 2)^2 + (z - 5)^2 = 11$, where $a > 2$.

The sphere K and the line l have exactly one common point, P . Find a and the coordinates of the point P .

Find an equation for the plane tangential to the sphere at the point P .

Answer to question 3

The plane α is determined by the direction vectors

$\vec{d}_1 = (4, -2, 5) - (1, 3, 6) = (3, -5, -1)$ and $\vec{d}_2 = (-1, 2, -1)$.

Thus a normal vector is $\vec{n} = \vec{d}_1 \times \vec{d}_2 = (7, 4, 1)$, and an

equation for α is: $7(x - 4) + 4(y - (-2)) + 1(z - 5) = 0$,

or $7x + 4y + z - 25 = 0$.

Category C0 (CAS-neutral)

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
20 -> va : 30 -> vb : 130 -> vc : 8 -> b
solve((c/sin(vc) = b/sin(vb) * c)
c = 12.2567110899
solve((a/sin(va) = b/sin(vb) * a)
a = 5.47232229321
solve(a/sin(va)=b/sin(vb),a)
MAIN DEG APPROX FUNC 3/30
    
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
12.256711089903 -> c 12.2567110899
solve(sin(va) = ch/b, ch)
ch = 2.73616114661
sqrt(b^2 + (c/2)^2 - 2*b*(c/2)*cos(va)
3.06861759674
2+(c/2)^2-2*b*(c/2)*cos(va)
MAIN DEG APPROX FUNC 3/30
    
```

Category C3

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
[4 -2 5] - [1 3 6] -> d1 [3 -5 -1]
[-1 2 -1] -> d2 [-1 2 -1]
crossP(d1, d2) -> n [7 4 1]
7*(x-4) + 4*(y-(-2)) + 1*(z-5) = 0
7*x + 4*y + z - 25 = 0
7*(x-4)+4*(y-(-2))+1*(z-5)=0
MAIN DEG AUTO FUNC 4/30
    
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
solve((x-a)^2 + (y+2)^2 + (z-5)^2 = 11, t)
t = sqrt(-5*a^2 + 28*a - 32 - a - 8) / 6 or t = -sqrt(-5)
|x=1-t and y=3+2*t and z=6-t
MAIN DEG AUTO FUNC 1/30
    
```


We now solve the equation of the sphere for t , using the three equations for the line as conditions.

The two answers for t are copied and stored in t_1 and t_2 .

Then equation $t_1 = t_2$ is solved for the parameter a .

Finally we solve the four equations, one for the sphere and three for the line, when $a = 4$. Note that there is only one answer. When the sphere and the plane have only one point P in common: $a = 4$ and $P(3, -1, 8)$.

Alternative method:

The centre of the sphere is $C(a, -2, 5)$.

With $P_0(1, 3, 6)$ on the line l , $\overrightarrow{P_0C} = \begin{pmatrix} a-1 \\ -5 \\ -1 \end{pmatrix}$,

and the distance from the centre C of the sphere to

the line l is $\text{dist}(C, l) = \frac{\left| \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} a-1 \\ -5 \\ -1 \end{pmatrix} \right|}{\sqrt{6}}$, where the

denominator is the norm of l 's direction vector :

$$\sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

The condition for a is that $\text{dist}(C, l)$ equals $\sqrt{11}$ which is the radius of the sphere. Once again we end up with $a = 4$, which is the same result as above.

This subquestion can be solved in several other ways.

The normal vector of the tangent plane at P is

$$\begin{pmatrix} 3-4 \\ -1-(-2) \\ 8-5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \text{ and the equation}$$

$$-1(x-3) + 1(y-(-1)) + 3(z-8) = 0, \text{ or}$$

$$-x + y + 3z - 20 = 0.$$

Question 4 (25%) [CAS version]

Consider a family of real functions

$f_a(x) = x - a \cdot \sqrt{x}$, $x \geq 0$, where a is a positive parameter.

Determine the zeros of f_2 , and the intervals where f_2 is decreasing or increasing.

Draw the graph of f_2 , and determine the range of f_2

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
t = 6 or t =
■ 8 → t1 : -[(-5·a² + 28·a - 32 + a + 8)] → t2
              6
              -[(-5·a² + 28·a - 32 + a + 8)]
              6
■ solve(t1 = t2, a) | a > 2 a = 4
solve(t1 = t2, a) | a > 2
MAIN DEG AUTO FUNC 3/30
  
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
■ solve((x-a)² + (y+2)² + (z-5)² = 11 and
t = -2 and x = 3 and y = -1 and z = 8
... 2*t and z = 6-t, {x, y, z, t} | a = 4
MAIN DEG AUTO FUNC 1/30
  
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
■ solve(norm(crossP([-1 2 -1], [a-1 -5
... 5, -11]))/sqrt(6)) = sqrt(11), a | a > 2
MAIN DEG AUTO FUNC 1/30
  
```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
■ [3 -1 8] - [4 -2 5] [-1 1 3]
■ -1·(x-3) + 1·(y+1) + 3·(z-8) = 0
-x + y + 3z - 20 = 0
-1*(x-3) + 1*(y+1) + 3*(z-8) = 0
MAIN DEG AUTO FUNC 2/30
  
```

Category C3

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6
■ factor(f2(x)) sqrt(x)·(sqrt(x)-2)
■ solve(f2(x) = 0, x) x = 4 or x = 0
■ d/dx(f2(x)) → df(x) Done
■ solve(df(x) = 0, x) x = 1
■ df(.5) -.414213562373
■ df(1.5) .183503419072
df(1.5)
MAIN DEG AUTO FUNC 2/30
  
```

A region M_a in the fourth quadrant is bounded by the graph of f_a and the x -axis. The area of M_a is A_a , and the volume of the solid of revolution generated by the rotation of M_a around the x -axis is V_a .

Calculate A_a and V_a .

Calculate a , such that $V_a = \pi \cdot A_a$.

Answer to question 4

The function f_2 is defined (barely visible on the screen shown in the right column). By factorising f_2 or by solving the equation $f_2(x) = 0$ we find the zeros of f_2 , namely $x = 0$ or $x = 4$.

We then differentiate f_2 , and denote the derivative by $df(x)$. The derivative is zero when $x = 1$.

We therefore calculate the values of the derivative at (e.g.) $x = 0.5$ and at $x = 1.5$, and find that to the left of $x = 1$ the derivative is negative and to the right it is positive.

Thus f_2 is decreasing when $0 \leq x \leq 1$ and increasing when $x \geq 1$.

The minimum is $f_2(1) = -1$. The factorisation of f_2 shows that $f_2(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Since f_2 is continuous, we can now conclude that the range of f_2 is the interval $[-1; \infty[$.

The factorisation of $f_a(x)$ shows that the graph of f_a has the same overall shape for all $a > 0$, and the region M_a is bounded by $x = 0$ and $x = a^2$.

The area A_a and the volume V_a are found by integration over the interval $[0; a^2]$. The answers are:

$$A_a = \frac{a^4}{6} \quad \text{and} \quad V_a = \pi \cdot \frac{a^6}{30}.$$

The volume is found by

multiplying the area by π when $a = \sqrt{5}$.

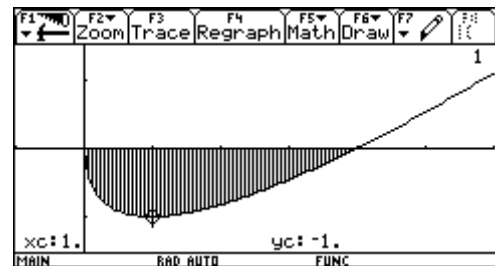
Question 5 (10 % , no special CAS version)

A manufacturer produces components for computers. The components are produced by machines A , B , and C , making 60%, 25%, and 15%, respectively, of the total production.

Experience shows that 20% of the components produced by A , 15% of the components produced by B , and 5% of the components produced by C , are defective.

Calculate the probability that a component, chosen at random, is defective. A component, chosen at random, appears to be defective. Calculate the probability that it was produced by machine C .

Answer to question 5 (see screen)



```

F1  F2  F3  F4  F5  F6  F7
Zoom Trace Regraph Math Draw

Define fa(x)=x-a*sqrt(x) Done
factor(fa(x)) sqrt(x)*(sqrt(x)-a)
solve(fa(x)=0,x)
x=a^2 and a>=0 or x=0
int_0^a^2 (0-fa(x))dx | a>0 a^4/6
solve(a^6*pi/30=pi*a^4/6,a)>0
  
```

```

F1  F2  F3  F4  F5  F6
Algebra Calc Other PrgmIO Clean Up
x=a^2 and a>=0 or x=0
int_0^a^2 (0-fa(x))dx | a>0 a^4/6
pi*int_0^a^2 ((fa(x))^2)dx | a>0 a^6*pi/30
solve(a^6*pi/30=pi*a^4/6,a)|a>0 a=sqrt(5)
solve(a^6*pi/30=pi*a^4/6,a)|a>0
  
```

Category C0 (CAS-neutral)

```

F1  F2  F3  F4  F5  F6
Algebra Calc Other PrgmIO Clean Up

.6*.2+.25*.15+.15*.05 .165
.15*.05 .045454545455
.165
.15*.05/.165
  
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Question 6 (15%)

[Extra CAS question in brackets]

A function f is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{\ln(y)}(x+2), \quad y > 1,$$

and the graph of f passes through the point $P(2, e)$.

Determine an equation of the line tangent to the graph of f at P . Determine the solution f , including the domain of f .

[Extra CAS: Show that the equation $f(x) = f'(x)$ has no solutions.]

Answer to question 6

The tangent line at $P(2,e)$ can be found before solving the differential equation.

The equation is $y = 4ex - 7e$.

Solving the differential equation using deSolve gives us an implicit solution.

When we solve for y we get two answers:

$$y = e^{\pm\sqrt{x^2+4x-11}}, \text{ with the condition that } x^2 + 4x > 11.$$

Since the graph passes through the point $P(2, e)$, and the range satisfies $y > 1$, we cannot use $y = e^{-\sqrt{x^2+4x-11}}$,

To find the domain we study the graph and the sign of $x^2 + 4x - 11$. We find that the sign is positive when $x < -2 - \sqrt{15} (\approx -5.87)$ or $x > -2 + \sqrt{15} (\approx 1.87)$.

Since the graph passes through the point $P(2, e)$ the domain interval must include $x = 2$ and the solution is:

$$y = e^{+\sqrt{x^2+4x-11}}, \quad x > -2 + \sqrt{15} (\approx 1.87).$$

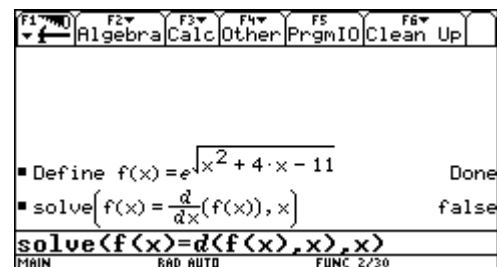
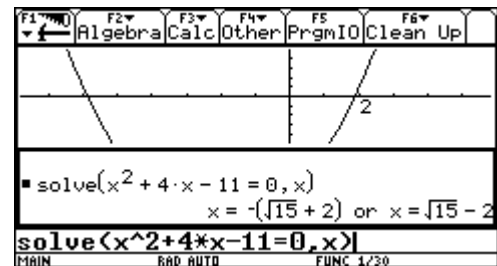
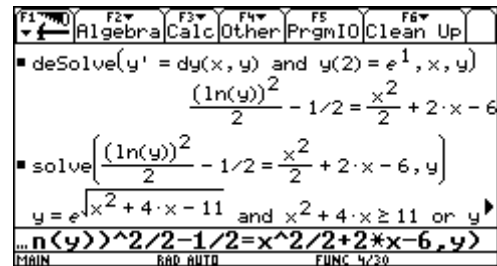
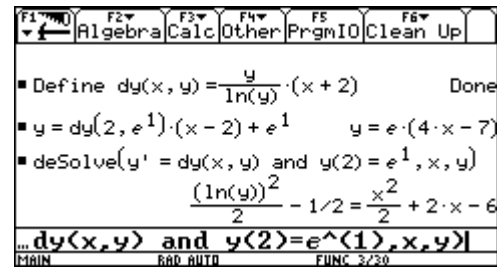
The last screen shows that the equation $f(x) = f'(x)$ has no solution for x . This can be seen in a different way, using PAP methods:

$$f(x) = f'(x) \Leftrightarrow f(x) = \frac{f(x)}{\ln(f(x))} \cdot (x+2)$$

$$\Leftrightarrow \ln(f(x)) = x+2 \Leftrightarrow \sqrt{x^2+4x-11} = x+2$$

$$\Leftrightarrow x^2+4x-11 = x^2+4x+4 \Leftrightarrow -11 = 4.$$

Categories C0, C1, C2



Question 7a (10%) [CAS version]

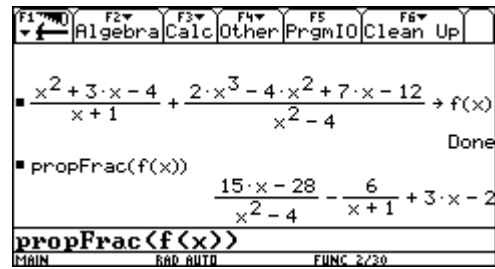
Category C3, C4

Consider the function

$$f(x) = \frac{x^2 + 3x - 4}{x + 1} + \frac{2x^3 - 4x^2 + 7x - 12}{x^2 - 4}$$

Find the domain of f .

Find an equation of each of the asymptotes.



Answer to question 7a

It is immediately apparent that the zeros of the denominators of f are -1, -2, and 2. Thus the domain is $\mathbb{R} \setminus \{-1, -2, 2\}$.

We expand the function into proper fractions. None of the three zeros of the denominators is also a zero of the numerator. Thus all three are vertical asymptotes.

The two fractional parts of $f(x)$ both tend to zero as $x \rightarrow \pm\infty$, giving the equation $y = 3x - 2$ as the oblique asymptote.

Conclusion

The graph has four asymptotes

$$x = -2, \quad x = -1, \quad x = 2, \quad \text{and} \quad y = 3x - 2.$$

Question 7b (10% , no special CAS version)

Category C0 (CAS neutral)

For the Galapagos Islands, the number of species of terrestrial plants on any island can be determined with good approximation as a function $N(x) = b \cdot x^a$ of the area x , in square miles.

It is given that $N(15) = 68$ and $N(174) = 149$.

Calculate a and b .



Consider two islands, where the area of one island is 2.5 times the area of the other island. Calculate the ratio between the number of terrestrial species on these two islands.



Answer to question 7b

The two given points are entered into lists, using the Stats/List Editor Application. By power regression (not all screens are shown in the right column) we find the function $N(x) \approx 28.6 \cdot x^{0.320}$.



By calculating $\frac{N(2.5 \cdot x)}{N(x)}$ we find that for a 2.5 times

larger island the number of species must be 1.34 times bigger.

6.2 Example of a final exam in Switzerland

In Switzerland every canton has its own educational system. Therefore no centralised final examinations exist, i.e. each teacher usually arranges his or her own examination. However, all examinations are published to assure quality. The following final exam is merely an example of many variants. Usually mathematics is tested with a written and an oral exam. The following is an example of an exam without an oral test. In lieu of the oral exam, there is a first part consisting of simpler traditional tasks that are completed without use of a calculator. For the sake of completeness this first part is included as well (without solutions).

6.2.1 1st part: Without calculator - Time 90 minutes

All of category C0 or C1

- (1) A sphere with centre $M = (2; 4; 0)$ has the point $A = (4; 5; -2)$ on its surface.
 - a) Determine the equation of the sphere.
 - b) B is the point on the spherical surface which lies nearest to the plane $E : x + 2y + 2z + 26 = 0$
Calculate the distance from B to E .
- (2) A circle is inscribed into a square with sides a . Now a square is inscribed into this circle etc. Determine the (infinite) sum of the areas of all circles.
- (3) $f_n(x) = n^2 \cdot x^3 - n \cdot x^2 + x$ ($n \in \mathbb{N}$)
 - a) Show that all inflection points W_n lie on a straight line.
Determine the equation of this straight line.
 - b) Examine whether the tangents of inflection are parallel, coincide or intersect.
- (4) Determine the area limited by the y -axis and the curves $y = \sin x$, $x \geq 0$ and $y = \cos x$, $x \geq 0$ (up to the first intersection point).
- (5) A ray of light coming from the point $A = (0; -2; 1)$ has the direction $\vec{a} = \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix}$ and is reflected at the plane $E : x + 3y + 2z - 24 = 0$.
 - a) Determine the point of reflection X .
 - b) Determine a parametric representation of the reflected ray of light.
- (6) A student receives a loan L . It is to be paid back in 8 equal annual payments and the first repayment is made 6 years after receipt of the loan. The interest rate is p %.
How large is a repayment?

6.2.2 2nd part: With CAS-Calculator - Time: 120 minutes

The first and second exercises are classical tasks. They could also be done by-hand, but it would be a time consuming effort. Exercises 3 and 4 require the use of a calculator. We will only give the solutions for these exercises. Since the development of such examination tasks is very important, we will show you the “history” of the development of exercise 4.

- (1) $P = (u; v)$ is an arbitrary point on the parabola with the equation

Category C1

$$y = -\frac{1}{2}x^2 + 2 \quad (-2 \leq x \leq 2).$$

- Determine P such that in the triangle ABP with $A = (-2; 0)$ and $B = (u; 0)$ the sum of the cathetes is maximal.
 - By rotating the triangle ABP around the x -axis a cone arises. Which is the maximum volume of such a cone?
 - The maximum cone from subtask b) is cut out from the rotation solid of the parabola (rotation around x -axis). What is the percentage of the waste that arises?
 - The normal to the parabola cuts the x -axis at point N . Determine P such that the triangle ANP has the greatest possible area. How big is the maximum area?
- (2) A roof of a tower with a square base has a shape like that shown in the figure to the right (length unit 1 m).

Category C1

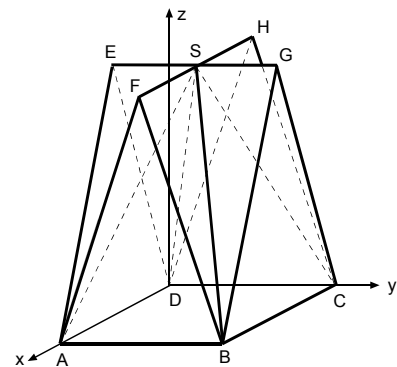
$$A = (4; 0; 0), B = (4; 4; 0), C = (0; 4; 0), D = (0; 0; 0)$$

$$E = (2; 0; 6), F = (4; 2; 6), G = (2; 4; 6), H = (0; 2; 6).$$

- a) Determine the coordinate equation of the plane E_1 that contains the points A, B and S .

Which is the angle α between the straight line DS and the plane E_1 ?

Which is the angle β between the planes E_1 and E_2 , where E_2 is the plane containing $BCHF$?



- One end of a 7 m long bar is attached in the centre of the segment \overline{DA} . It is perpendicular to the plane E_2 from subtask a) and protrudes through the gable area BCG . Can this bar be used as a flag bar, if at least 2.75 m is necessary to attach the flag?
- A sphere is fastened to a chain at the end of the bar from subtask b). What is the largest radius the ball may have so that it does not touch the gable area BCG .

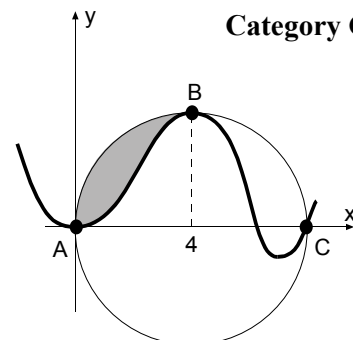
a) and b) from “Thilo Vohrer: Abitur Prüfungsaufgaben 1994-2001. ISBN 3-89496-200-3

- (3) a) Determine a polynomial f with smallest possible degree, which satisfies the following conditions:

The graph (which does not have to correspond to the accompanying illustration)

- has a horizontal tangent at the origin,
- touches the circle k with radius 4 and centre $(4; 0)$ at B ,
- has an inflection point at C .
- The shaded area amounts to 6 (integral between A and B).

Category C3



b) Outline the real curve as well as the circle.

Tip: Accelerate the calculation by using a numeric value in the function.

c) Choose a parameter k (instead of 6) for the integral between A and B . Can k be arbitrary? (Give a reason for your answer.) If the answer is yes, outline the curve for $k = 0$ (including the circle).

d) Discuss the asymptotic behaviour for $x \rightarrow \pm\infty$ for allowable values of k .

(4) An aeroplane flies on a greater circle at 10 km altitude around the earth. **Category C3**
It passes Pyongyang in North Korea (126° east; 39° north) and Khartoum in Sudan (33° east; 16° north). Earth radius $r = 6371$ km.

a) The aeroplane needs 20 hours from Pyongyang to Khartoum. What is its average speed?

b) At which point (geographical length) does the plane cross the equator for the first time after flying past Khartoum?

c) What is the northern-most point of the flight path (Cartesian coordinates are sufficient)?

d) Does Iraq lie on this greater circle?

Iraq extends from latitude 27° to 37.5° north on the meridian 44° east through Baghdad.

6.2.3 Credits for the exam

First part: 30 points (5 points per task)

Second part: 40 points (10 points per task).

Altogether: 70 points altogether.

Highest mark for 45 points or more. The pass mark was 23 points. Average was 37.3 points.

6.2.4 Solution to task 3 (2nd part)

Solution to a) and b):

Establish the conditions by-hand:

$$f(0) = 0, f'(0), f(4) = 4, f'(4), f(8) = 0, f'(8), A = 4\pi - \int_0^4 f(x)dx = 6.$$

Determine the polynomial due to the number of conditions (here 7):

$$f(x) = a \cdot x^6 + b \cdot x^5 + c \cdot x^4 + d \cdot x^3 + e \cdot x^2$$

Please let the students find the degree of the polynomials, because it works in any case, but the result will be wrong and the graph provides them with an immediate self-check.

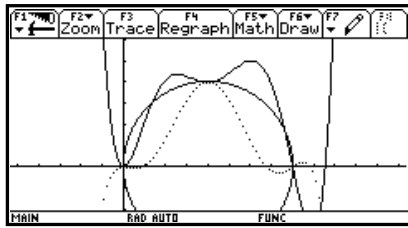
To save calculation time the last two constants were set to zero (according to the first two conditions). Now we determine the first and second derivatives and store them in $f1(x)$ and $f2(x)$. We store the remaining five equations above in $e1$, $e2$, $e3$, $e4$ and $e5$ and solve the corresponding system of equations: solve ($e1$ and $e2$ and $e3$ and $e4$ and $e5$, $\{a, b, c, d, e\}$).

The result appears in the form $a = \dots$ and $b = \dots$ and $\dots e = \dots$ We store $f(x)$ | result in $y1(x)$ and we get (numeric values to accelerate the drawing process)

$$y1(x) = -0.001x^6 + 0.036x^5 - 0.300x^4 + 0.930x^3 - 0.587x^2.$$

Now draw the function and the circle (as a two-part function or approximately with F7 Circle in the graph window, or with Circle 4,0,4 in the Home Screen).

Solution to c):



Modify the last equation and replace 6 by k and solve the system of equations again (all taken from history). We see that in the new function k appears in linear terms only and therefore the system is always solvable.

Store the function in $y2(x)$ and draw it for $k = 0$ (see figure, dotted line for $k = 6$).

Solution to d):

The asymptotic behaviour depends on the sign of the highest coefficient x^6 .

The coefficient of x^6 is $a = -0.000657(k-3.766)$, thus

$$k > 3.766 \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) = -\infty$$

$$k < 3.766 \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) = +\infty$$

Special case: $k = 0 \Rightarrow$ Coefficient of $x^5 < 0 \Rightarrow \lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$

Student solution to task 3

Kantonsschule Heerbrugg
Maturitätsprüfungen
Klasse: M Na Ta

Name und Vorname: [REDACTED]
Fach: Mathematik

3)

a) Bedingungen:

1. $f(0) = 0 \checkmark \Rightarrow g = 0$
2. $f'(0) = 0 \checkmark \Rightarrow f = 0$
3. $f(4) = 4 \checkmark$
4. $f'(4) = 0 \checkmark$
5. $f''(8) = 0 \checkmark$
6. $f''(0) = 0 \checkmark$
7. $\int_0^4 f(x) \cdot dx = 6 \checkmark$

\Rightarrow Polynom 6. Grades: $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

\Rightarrow TR: $f(x) = -0.001468x^6 + 0.035892x^5 - 0.3x^4 + 0.92x^3 + 0.5873x^2$

b)

c)

$$a = -0.000657(k-3.7664)$$

$$b = 0.01665(k-3.74457)$$

$$c = -0.1107(k-4.00823)$$

$$d = 0.5749(k-4.3824)$$

$$e = -0.785256(k-5.25208)$$

Wird k kann zwischen 0 und 4π liegen.

Negativ darf k nicht sein, sonst geht die Kurve nicht durch B!
 $k=0$

d) Parameter a (bei höchster Potenz) $a = -0.000657(k-3.7664)$

Wenn $k < 3.7664 \Rightarrow x \rightarrow \infty : y \rightarrow -\infty$
 $x \rightarrow -\infty : y \rightarrow +\infty \checkmark$

Wenn $k > 3.7664 \Rightarrow x \rightarrow \infty : y \rightarrow -\infty$
 $x \rightarrow -\infty : y \rightarrow +\infty \checkmark$

Wenn $k = 3.7664, a = 0 \Rightarrow x \rightarrow \infty : y \rightarrow -\infty$
 $x \rightarrow -\infty : y \rightarrow +\infty \checkmark$
 x fällt weg.

9/12

Introduction of parameters leads to a new dynamic in algebra. With computer programs you can vary the parameter k with sliders and change the graph of the function interactively. In geometry this interactivity is well known through dynamic geometry programs. Of course, the size of the area is not the only number, which can be parameterised. In each equation numbers can be replaced by parameters. By doing so you get unlimited possibilities for individual and group activities and their presentation in the form of animations.

6.2.5 Solution to task 4 (2nd part)

- a) Store the polar coordinates as a (vector) function of three variables, namely radius r , degree of longitude x and degree of latitude y :

$$\vec{v}(r, x, y) = \begin{pmatrix} r \cos(x) \cos(y) \\ r \sin(x) \cos(y) \\ r \sin(y) \end{pmatrix}.$$

With CAS: $[r \cos(x) \cos(y), r \sin(x) \cos(y), r \sin(y)] \text{ [STO]} \vec{v}(r, x, y)$.

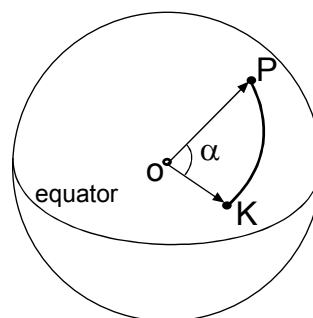
Define the position vectors for Pyongyang P and Khartoum K . To avoid large numbers we take $r = 1$ (unit sphere).

$$\overline{OP} \parallel \vec{p} = v(1, 126^\circ, 39^\circ) \quad \overline{OK} \parallel \vec{k} = v(1, 33^\circ, 16^\circ)$$

Store these vectors as numerical values. Switch now to radian mode and calculate the angle between \vec{p} and \vec{k}

$$\alpha = \arccos \frac{\vec{p} \cdot \vec{k}}{|\vec{p}| |\vec{k}|} = 1.436$$

$$\text{Hence } v = \frac{s}{t} = \frac{6381 \cdot \alpha}{20} = 458.163 \frac{\text{km}}{\text{h}}.$$



- b) *Students think differently when solving this problem with CAS. By-hand students fear the cross product like the devil the holy water, because it requires much calculation. To find a vector for a certain direction you have to find two vectors perpendicular to it and CAS calculates the cross product in no time. Students do the more important mathematical part, CAS the calculation part.*

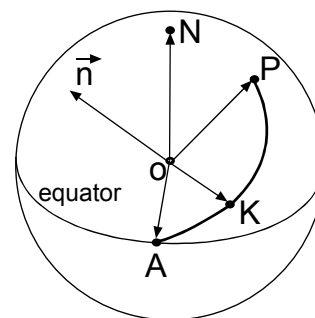
Let $\vec{n} = \overline{OK} \times \overline{OP}$ be a normal vector to the greater circle through Pyongyang and Khartoum and A the intersection point of this circle with the equatorial circle. Then we have the following conditions:

$$\overline{OA} \perp \vec{n} \text{ and } \overline{OA} \perp \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \overline{OA} \parallel \vec{e}_3 \times \vec{n}$$

If we take the unit vector $\vec{e}_A = \frac{\overline{OA}}{|\overline{OA}|}$, its first component will

just be the cosine of the angle sought. Therefore we get the longitude of point A (switch to degree mode):

$$\varphi = \arccos(\text{1st component of } \vec{e}_A) \stackrel{\text{CAS}}{=} \arccos(\vec{e}_A[1, 1]) = 13.854^\circ$$



- c) Let H be the highest (northern-most) point of the flight path. We have the following conditions:
 $\overline{OH} \perp \vec{n}$ and $\overline{OH} \perp \overline{OA} = k\vec{e}_A \Rightarrow \vec{h} = \vec{n} \times \vec{e}_A$ is a vector in direction of \overline{OH} .

$$\text{Hence } \overline{OH} = 6381 \frac{\vec{h}}{|\vec{h}|} = \begin{pmatrix} -1150.281 \\ 4664.148 \\ 4199.968 \end{pmatrix}.$$

- d) Store the unit vector $\vec{i} = v(1, 44^\circ, x)$ of the Iraqi meridian.

The condition for the angle x to the greater circle is $\vec{n} \times \vec{i} = 0$.

This equation is solved with respect to x and results in: $x = 180^\circ \cdot (k + 0.1317)$.

For $k = 0$ we get $x = 23.705^\circ$, which means that the greater circle passes south of Iraq.

Using defined functions as a module makes CAS-calculators very efficient and encourages the tackling of realistic problems. Both of the above tasks would be very difficult to solve by-hand or with scientific calculators. Tasks of this kind look very difficult at first glance, but with some training they become standard problems for students.

6.2.6 “History” of task 4 (2nd part)

It is certainly of interest to know which exercises and examinations preceded this final exam. Therefore we will list all examples, exercises and exam questions completed in preparation for task 4. The subject was dealt with approximately seven months before the final exam. The solutions are similar to those of task 4 and thus are omitted. The beginning of the instructional unit included the theory of the cross product with some standard examples without use of polar coordinates and often solved by-hand. The following exercises are hardly solvable by-hand. With the time saved CAS students can solve many more problems about the cross product. (On the other hand tasks about intersection of planes and lines become less important.) The main idea is as mentioned above: if we search for a vector, we have to find two vectors perpendicular to this vector.

Exercise 1 (without polar coordinates)

Category C1

Given two vertices $A = (4; 0; 1)$, $B = (0; 2; 7)$ of a tetrahedron $ABCD$ and a point $P = (1; 1; 1)$ in the plane ABC .

- Determine the vertices C and D of a regular tetrahedron $ABCD$. Calculate just one solution and give the number of possible solutions.
- Determine the vertex E of a tetrahedron $ABCE$ with an equilateral triangle ABC such that the height through E passes the centroid of ABC and E lies in the x, y -plane.

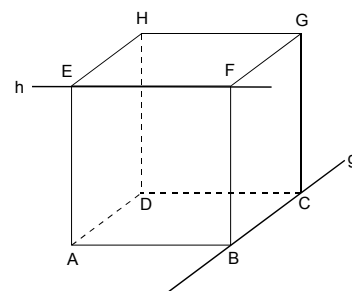
Exercise 2 (exam question)

Category C1

Given straight lines

$$g: \vec{r}_1 = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + t \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} \text{ and } h: \vec{r}_2 = \begin{pmatrix} 8 \\ 0 \\ 12 \end{pmatrix} + s \begin{pmatrix} 3 \\ -12 \\ 4 \end{pmatrix}$$

- Show that g and h are (skewed) perpendicular.
- Calculate the vertices A, B, F of a cube with edges B, C on g and E, F on h



Exercise 3**Category C1**

An unidentified flying object (UFO) passes the earth on a straight line at constant velocity. It has been observed by a radar control perpendicular above Saint Gall ($9^{\circ}22'$ east; $47^{\circ}22'$ north) at an altitude of $50'000$ km.

Two hours later the UFO appears perpendicular above New York (74° west; 41° north) at a height of $30'000$ km.

- What is the velocity of the UFO? (Don't consider the rotation of the earth for the moment. Earth radius 6371 km.)
- Solve a) but take into consideration the rotation of the earth.
- What is the angle between the flight direction and the tangent to the meridian direction above Saint Gall?
- When is the UFO nearest to the earth (time after passing Saint Gall)?

Exercise 4**Category C1**

An ocean liner starts in Cape Town (18.5° east; 34° south) sailing towards New York (74° west; 41° north) using the shortest route possible (earth radius 6371 km).

- How many days does the ocean liner need for the journey if its average velocity is 36 km/h?
- When does the ocean liner cross the equator?
- Why is the time needed from Cape Town to the equator, subsequently to New York, not proportional to the corresponding latitudes?
- (encroachment on other subjects) Is the shortest route from Cape Town to New York possible at all (on a globe or an atlas)? Compare the latitude and climate of Cape Town with a similar town in the northern hemisphere.

Exercise 5 (exam question)**Category C1**

A meteorite is observed perpendicularly above Zurich (24° east; 47° north) at a height of $60'000$ km. Five hours later the meteorite appears at a height of 1000 km perpendicularly above the north pole (uniform continuous movement, earth radius 6371 km).

- What is the velocity of the meteorite?
- Determine the position vector, dependant on the time (Zurich: $t = 0$).
- Will the meteorite strike the earth? If the answer is yes, where will it strike (Cartesian coordinates), otherwise what is its closest distance to the earth?

6.3 Final exam 2001 in St. Pölten, Austria (C3/C4)

As many non Austrian readers will know, we do not have national exams in Austria. Each secondary school and in some schools each individual teacher, is responsible for setting up tasks for the final exams (= "Matura". In most schools teachers agree on one common set of problems. For example, in my school there were three teachers teaching senior mathematics during 2001, namely Tania Koller, Günter Schraik and myself. We decided to give three tasks: one from statistics with some probability theory, one from financial mathematics (investment calculation) and one from analysis. The working time for the three tasks was 4 hours without a break.

It was my responsibility to "design" the analysis problem. The problem was inspired by Carl Leinbach's lecture in Gösing and by a problem set by Sibylle Stachniss-Carp. The problem is given below:



1 cm = 600m

A 18-hole golf course is planned for the region north of the River Danube. The northern border of the golf area will be formed by a newly built street which shall be connected near *Winkl* (point **W**) to the straight street S_1 which will also be constructed. The new street shall pass point **O** to finally run near *Absdorf* (point **A**) into the proposed road, S_2 .

- a) As a first approximation find the line of construction of the northern border, which shall be described by a polynomial function of lowest degree. The connections between this road and S_1 and S_2 must be smooth.

To assist in finding the function, read off values from the map as accurately as possible. What is the term of the approximating polynomial function?

Give reasons for your choice of the polynomial degree.

- b) The connection between roads should not be only smooth (i.e. differentiable), but should also avoid sudden changes of curvature. In order to achieve this goal the second derivatives for the curves must be common.

Find the polynomial function which will describe the run of the road in this case and use this function for the following tasks.

- c) Sketch the new road on the map using at least 6 additional points to those given in the original problem. Define the line (S_1 - new road - S_2) as a piecewise defined function. Write this using correct syntax.
- d) What is the difference in the lengths of golf course obtained from each model?
- e) What area is available for the course if the southern border is formed by the existing road connecting W and A? In first approximation you can use the airline between W and A (the shortest connection between A and B).
- f) Discuss whether the real area would be greater or less that the calculated value. Try to improve your model.. Briefly describe your reasons for choosing your new model.

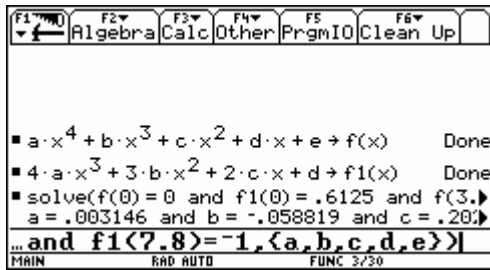
6.3.1 Proposal for a solution

- a) Most students chose the origin at W with x - axis horizontal and y -axis vertical, but there were also some who chose line WA as the x -axis with W as the origin. We will consider the first model:

Using measurement we find the coordinates: $W(0 | 0)$, $O(3.9 | 2.7)$ and $A(7.8 | 0.8)$. The slope of S_1 is either found by rise/run ≈ 0.6125 or $\tan(31^\circ)$. To model the smooth connection at A one has to draw the tangent on S_2 using a ruler and read off the slope as ≈ -1 .

We now have enough information to set up 5 equations to determine 5 variables, which leads to the conclusion to use a polynomial of degree 4: $y(x) = a x^4 + b x^3 + c x^2 + d x + e$.

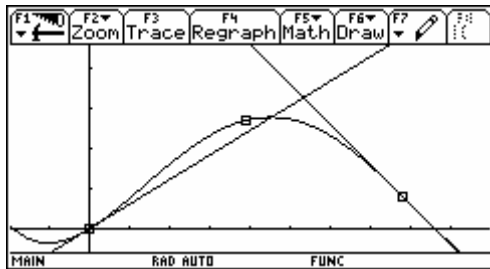
$$\begin{aligned}
 y(0) &= 0 && \rightarrow && e = 0 \\
 y'(0) &= 0.6125 && \rightarrow && d = 0.6125 \\
 y(3.9) &= 2.7 && \rightarrow && 3.9^4 a + \dots = 2.7 \\
 y(7.8) &= 0.8 && \rightarrow && 7.8^4 a + \dots = 0.8 \\
 y'(7.8) &= -1 && \rightarrow && 4 \cdot 7.8^3 + \dots = -1
 \end{aligned}$$



The solution of the system of linear equations was obtained by using the `rref()` command for producing the row reduced echelon form.

The new generation of calculators offers an easier and more familiar way to solve simultaneous equations.

$$y(x) = 0.0031460x^4 - 0.0588186x^3 + 0.2020053x^2 + 0.6125x \text{ (stored as po14(x))}$$



The figure shows the graph of the polynomial together with the two tangents.

- b) Two additional conditions give more equations. Two more parameters mean we can now consider a polynomial of degree 6. At the connecting points the functions must have a common second derivative.

In point W: $y''(0) = 0$

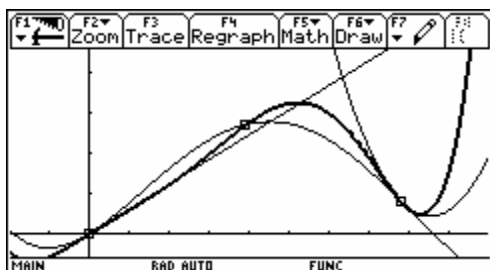
To find an approximation for the second derivative at point A we need a quadratic approximation for S_2 at point A. We read off three points on S_2 close to and including A: A(7.8 | 0.8), B(8.3 | 0.45), C(8.9 | 0.5).

Using the regression tool, or solving a system of equations, we find the parabola passing through A, B and C: $p(x) = 0.7121x^2 - 12.1652x + 52.3627$

Hence $y''(7.8) = p''(7.8) \approx 1.4242$ and we improve $y'(7.8) = p'(7.8) \approx -1.056$.

The 7×8 matrix must be filled and using `rref()` again we find the solutions for the parameters a through g in order to define the polynomial of degree 6:

$$y = 0.000309x^6 - 0.005205x^5 + 0.0245x^4 - 0.02947x^3 + 0.6125x \text{ (stored as po16(x))}$$



The figure shows both polynomials together with the graphic representation of S_1 and the approximating parabola of S_2 .

The piecewise definition of the course is: when $(x < 0, 0.6x, \text{when}(x < 7.8, \text{po16}(x), p(x)))$.

Line WA: $y = \frac{0.8}{7.8}x$ (stored as `ger(x)`).

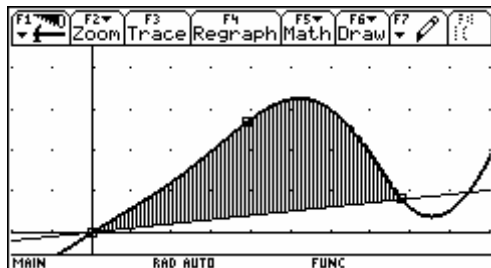
c) From the table we can easily obtain the coordinates of points of the curve to transfer them onto the map.

d) The arc lengths can be found in the Graph Screen (F5) or in the Home Screen:

The lengths are 5.50 km and 5.95 km, so there is a difference of approximately 450 m.

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
DATA		x-Werte	y-Werte			
	c6	c7	c8	c9		
1	0	1	.6026	0		
2	.8000	2	1.2345	4.7000		
3		3	1.9868	7.8000		
4		4	2.7717			
5		5	3.2538			
6		6	3.0041			
7		7	1.8769			

c7, Title="x-Werte"



F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
3.09e-4 · x ⁶ - .005205 · x ⁵ + .0245 · x ⁴ - .02					
Done					
.8					
7.8 · x → ger(x) Done					
arcLen(pol4(x), x, 0, 7.8) 9.1721					
arcLen(pol6(x), x, 0, 7.8) 9.9194					
∫ ₀ ^{7.8} (pol6(x) - ger(x)) dx 11.8917					

f(pol6(x)-ger(x), x, 0, 7.8)

e) The area is the integral shown above. But, not surprising for all of you who are teachers is that the most difficult problem for the students was the conversion of the 11.89 cm² into km²!!!

$$\text{Area} \approx 11.89 \times (600 \text{ m})^2 = 11.89 \times (0.6 \text{ km})^2 = 4.3 \text{ km}^2.$$

f) The real area is larger.

We can approximate the additional area by a triangle $\Delta: 8 \times 0.6 / 2 = 2.4 \text{ cm}^2 \rightarrow 0.93 \text{ km}^2$ (total 5.23 km²). Or we find another parabola passing through W, A and point D(4.7 | -0.1) and then use the TI to calculate the area between pol6(x) and this parabola giving a total sum of 5.17 km².

The background of this problem is to find a polynomial function using given properties. This version makes the problem more open because of the given data and the fact that students must fix their own system of coordinates..

The system of simultaneous equations is difficult for students to handle without technology.

The very useful and important concept of piecewise defined functions is used. My experience was that the students found a lot of ways to estimate the additional area (f) and demonstrated their understanding of the basics of integrating (summing up smaller areas).

A nice story is the following: This example was one out of three and the working time was 4 hours. Some days after the exam when the students asked for their results one student came to us and asked: "May I have a copy of this street-example, please? I liked it so much and I enjoyed the examination, especially this problem. I'd like to keep it as a memory of my maths-end examination". I believe that this experience does not need any additional comment.

In the following I present parts of students' solutions.

6.3.2 Solution Student 1

Student 1 had horizontal/vertical axes and grabbed the data out of the map.

1) $W = y = 5,8 \text{ cm}$ $x = 0$
 $W(0) = 0,6$ ✓ $W'(0) = 0,6$ ✓
 $O(4/8,4)$ ✓ $A'(7,9) = -1$ ✓
 $A(7,9/6,6)$ ✓
ending: $x=1$ $y=0,6 =$ ~~Ans~~ $\text{Ans} = 0,6$
 $\text{Ans} = -1$

For question b) he noted: *Polynomial function of degree 6, because we have 7 pairs of data.*

1. Ableitung: $6ax^5 + 5bx^4 + 4cx^3 + 3dx^2 + 2ex + f$
2. Ableitung: $30a^4x^4 + 20bx^3 + 12cx^2 + 6dx + 2e$
 \rightarrow rref (matrA1) $= a = 0,002$ $b = -0,047$
 $c = 0,367$ $d = -0,96$ $e = 0$ $f = 0,6$ $g = 5,8$
 $\Rightarrow 0,002x^6 - 0,047x^5 + 0,367x^4 - 0,96x^3 + 0,6x + 5,8$

6.3.2 Solution Student 2

answered for f): *The real area is larger. In order to improve the estimation I'd inscribe triangles between the road and line WO and then calculate the sum of the areas".*

However in the exam he did not perform this calculation.

6.3.3 Solution Student 3 (bright female)

c) Punkt 1: $y(1) = 0,64$
2: $y(2) = 1,41$
3: $y(3) = 2,21$
4: $y(4,8) = 2,92$
5: $y(5,8) = 2,59$
6: $y(6,8) = 1,75$
d) Funktion der Luftlinie:
 $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$
 $y - 0 = \frac{2,8 - 0}{7,8 - 0} (x - 0)$
 $y = \frac{1}{39} x \rightarrow g(x)$
Fläche: $A = \int_0^{7,8} f(x) - g(x) dx = 11,128 \text{ cm}^2 \approx 668 \text{ m}^2$

e) Die tatsächliche Fläche ist größer.

einige weitere Punkte von ... Straße ablesen → in ein
 Datenblatt schreiben → Regressionslinie durchlegen → mit dieser
 neuen Funktion f. d. Straße rechnen. weil, da ein
quad. Typ
oder hier

$$A = \int_0^{7,8} f(x) - g(x) dx = 14,015 \text{ cm}^2 \approx \underline{847 \text{ m}^2}$$

76

Translation of her comment:

e) The area is greater.

Read off some points from the road → edit a data sheet → Regression line → work with this function.

She applied only linear regression (I had expected to use at least quadratic regression because of the course of the street.) and she had problems converting the calculated area of the map into km².

6.3.4 Solution Student 4

2) Polynomfunktion 4. Grades

$$y = ax^4 + bx^3 + cx^2 + dx + e \rightarrow \text{quer}$$

$$y' = 4ax^3 + 3bx^2 + 2cx + d \rightarrow \text{quer}$$

$V(0|0)$

$0(4,2|2,3)$

$1(7,9|0)$

$k_{4,2} = 0,5$

$k_{7,9} = -1,1$

$y(0) = 0 \checkmark$

~~$y(1) = 2,3$~~

$y(4,2) = 2,3 \checkmark$

$y(7,9) = 0 \checkmark$

$y'(0) = 0,5 \checkmark$

$y'(7,9) = -1,1 \checkmark$

für Notizen vorbereiten
 ordnen man die
 x-Wert in
 que bzw. quer
 umstellt

He had the good idea to take WA as x-axis.

He noted the values for the matrix and found the coefficients of the quartic. He used a similar approach for the polynomial of degree 6.

Then you can see how he found the approximation for the parabola to derive the slope at point A.

Finally I present his sketch of the course for the street. As an extra he calculated the turning point of the newly created street.

Knoten	a	b	c	d	e	
	0	0	0	20	21	0
34,16%	79,083	11,640	4,2000	1	2,5	
38,5,00%	49,0539	62,7100	7,9000	1	0	
0	0	0	1	0	0,5	
1972,150	177,200	15,9300	1	0	-1,1	
spektrum unter a und						270
ref (0,50)	$a = 0,0029$		$c = 0,1907$		$e = 0$	
	$b = -0,0597$		$d = 0,6$			

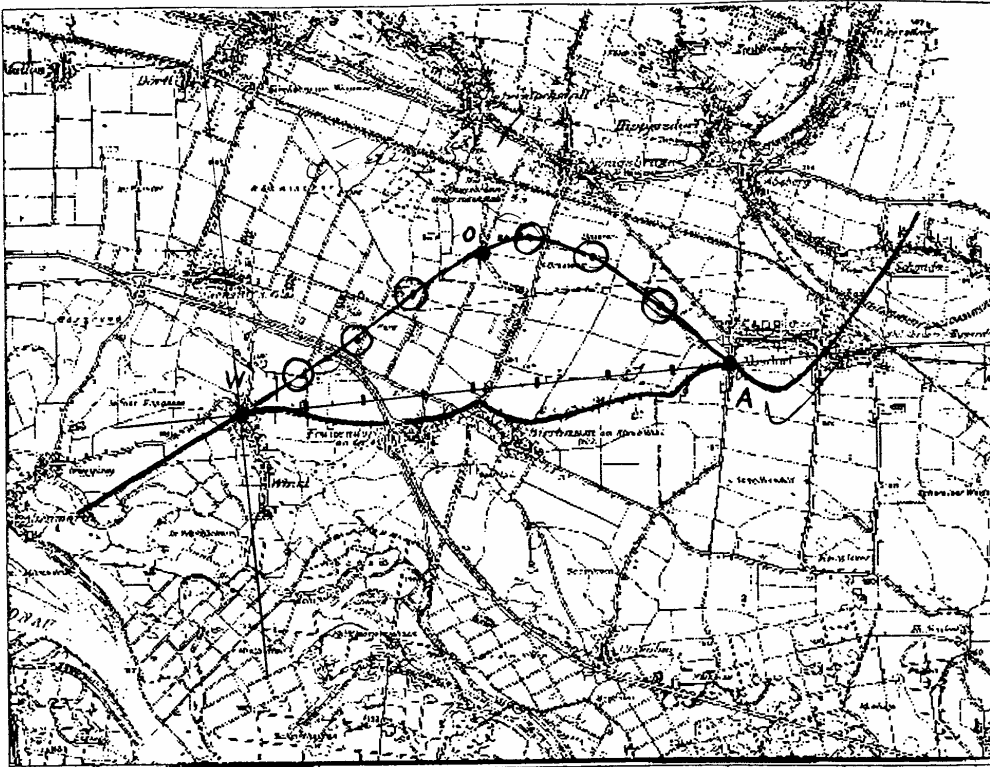
Punkt auf d. 2. Kurve

$A (2, 10)$
 $A_1 (8,85 | 0,45)$ → data → quadratische Pol
 $A_2 (1,52 | 0,25)$ ↙ Lösung

Regressionslinie
 $a = 0,27$
 $b = -5,305$ → $y_1(x)$
 $c = 8829,57$

Steigung in Punkt a ermitteln
~~→~~ → $d(d(y_1(x), x), z)$
 → 0,65586

1



75

Vorlage für 1

$y(110,515)$	$y(311,7115)$	$y(2,9057)$	} $y(71,07)$
$y(211,098)$	$y(412,2325)$	$y(612,8334)$	

Golfkurs: $(4,8685/2,9254)$

For answering the question about the southern part of the area he wanted to perform a numerical integration applying the trapezoidal sum, but made a calculation error.

Finally a nice story.

Two weeks after the end examination the students came to school to get some information about their marks. One of them asked for a copy of the "Golf Course Task" because she "had so much fun at the examination in solving this problem". No comment necessary!

Reference: Neue Aufgaben für das Unterrichten mit DERIVE & TI-89/92/92+Voyage 200, Vol. 1 with friendly permission of bk-teachware.

6.4 Belgian entrance examinations for civil engineering students

There are no central examinations in secondary schools in Belgium. The following questions are examples from the entrance examinations at three Flemish Universities (Louvain, Brussels and Ghent - for the last time organized in 2003), for students who want to commence studies in civil engineering in the Faculty of Applied Sciences. In the examinations the students are allowed to use the tools they learned to work with at secondary school, including CAS. The questions are not representative for Belgium because only approximately 1000 students (mostly good in math) attempt the examination each year.

Question 1 - Category C3

Consider the following system of linear equations in the real variables x, y, z :

$$\begin{cases} a \cdot x + y + z = 1 \\ x + a \cdot y + z = a \\ x + y + a \cdot z = a^2 \end{cases}$$

For which value(s) of the real constant a does the system have

- No solutions?
- Exactly one solution? Find this solution.
- Infinitely many solutions? Find the solutions.

Solution

First of all, be aware of the fact that a CAS cancels symbolic factors in the numerator and denominator, without mentioning that the symbolic factor may not be zero!

To detect the different cases, first calculate the zeros of the determinant of the matrix A of the system.

Case 1 $\det(A) \neq 0$: $a \neq 1$ and $a \neq -2$

The system has exactly one solution. The solution can be found with the reduced row-echelon form of the augmented matrix of the system.

A screenshot of a CAS interface showing the input of the system of equations. The top menu bar includes F1 Algebra, F2 Calc, F3 Other, F4 PrgmIO, F5 Clean Up, and F6 Up. The input area shows the equations: $\frac{a}{a}$ (with a cursor) and $\frac{a^2 - b^2}{a - b}$ (with a cursor). The bottom status bar shows MAIN, RAD AUTO, and FUNC 2/30.

A screenshot of a CAS interface showing the calculation of the determinant and solving for a . The input is $\det\left(\begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}\right)$, which results in $(a-1) \cdot (a^2 + a - 2)$. Below this, the input is $\text{solve}((a-1) \cdot (a^2 + a - 2) = 0, a)$, which results in $a = 1$ or $a = -2$. The bottom status bar shows MAIN, RAD AUTO, and FUNC 2/30.

A screenshot of a CAS interface showing the augmented matrix and its input. The input is $\begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{bmatrix} \rightarrow m$. The bottom status bar shows MAIN, RAD AUTO, and FUNC 1/30.

A screenshot of a CAS interface showing the reduced row-echelon form of the augmented matrix. The input is $\text{rref}(m)$. The result is $\begin{bmatrix} 1 & 1 & a & a^2 \\ 1 & 0 & 0 & \frac{1}{a+2} - 1 \\ 0 & 1 & 0 & \frac{1}{a+2} \\ 0 & 0 & 1 & \frac{(a+1)^2}{a+2} \end{bmatrix}$. The bottom status bar shows MAIN, RAD AUTO, and FUNC 2/30.

The solution is $x = \frac{1}{a+2} - 1$, $y = \frac{1}{a+2}$, $z = \frac{(a+1)^2}{a+2}$.

Note The special case $a = 1$ is not detected by row reduction as the factor $a - 1$ is cancelled during the reduction process!

Case 2 $\det(A) = 0$: $a = 1$ or $a = -2$

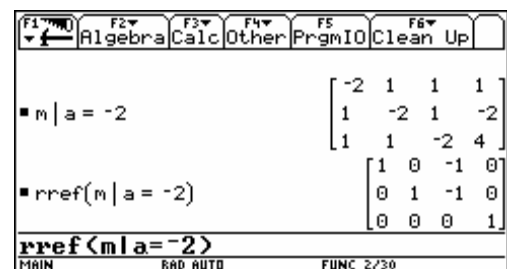
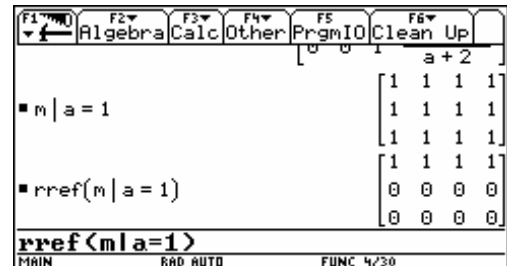
(i) $a = 1$

The system reduces to $x + y + z = 1$ and has infinitely many solutions:

$$\begin{cases} x = 1 - s - t \\ y = s \\ z = t \end{cases} \quad (s, t \in \mathbb{R})$$

(ii) $a = -2$

The last equation of the reduced system is $0 \cdot x + 0 \cdot y + 0 \cdot z = 1$. The system has no solutions.



Analogous exercise

Discuss the system $\begin{cases} a \cdot x + a \cdot y + b \cdot z = 1 \\ a \cdot x + 2y + b \cdot z = 1 \\ b \cdot x + b \cdot y + a \cdot z = 1 \end{cases}$ (unknowns x, y, z and parameters a, b).

Question 2 - Category C0

The population migration between two geographical regions in a city, the East and the West, is as follows. Each year, 30% of the population in the East migrates to the West, while only 10% of the West moves to the East. (Neglect birth, mortality and migration to other regions).

- Give a matrix description of this situation.
- At the end of the year 2000, the East has 600000 inhabitants and the West has 400000 inhabitants. Use the matrix description to find the number of inhabitants in the East and the West in 2005 and 1999.
- If this pattern continues, find the limit situation of the population distribution in the East and the West.

Solution

a) Let x_i be the number of inhabitants in the East and y_i the number of inhabitants in the West at the end of year i , then the number of inhabitants at the end of year $i + 1$ becomes:

$$\begin{cases} x_{i+1} = x_i - 0.3x_i + 0.1y_i \\ y_{i+1} = y_i + 0.3x_i - 0.1y_i \end{cases} \quad \text{or} \quad \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = A \cdot \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \quad (\text{the transition matrix}).$$

$$b) \begin{bmatrix} x_{2005} \\ y_{2005} \end{bmatrix} = A \cdot \begin{bmatrix} x_{2004} \\ y_{2004} \end{bmatrix} = A^2 \cdot \begin{bmatrix} x_{2003} \\ y_{2003} \end{bmatrix} = \dots = A^5 \begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix}$$

$$\begin{bmatrix} x_{2005} \\ y_{2005} \end{bmatrix} = A^5 \cdot \begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix} = \begin{bmatrix} 0.308 & 0.231 \\ 0.692 & 0.769 \end{bmatrix} \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix} = \begin{bmatrix} 277216 \\ 722784 \end{bmatrix}$$

$$\begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix} = A \cdot \begin{bmatrix} x_{1999} \\ y_{1999} \end{bmatrix}$$

$$\begin{bmatrix} x_{1999} \\ y_{1999} \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix} = \begin{bmatrix} 1.5 & -0.17 \\ -0.5 & 1.17 \end{bmatrix} \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix} = \begin{bmatrix} 833333 \\ 166667 \end{bmatrix}$$

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
$\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix} \rightarrow a$					
a^5					
$a^5 \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix}$					
$a^5 * [600000; 400000]$					

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
a					
a^{-1}					
$a^{-1} \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix}$					
$a^{-1} * [600000; 400000]$					

c) Observe the powers of the transition matrix A :

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
a^{10}					
a^{15}					
a^{20}					

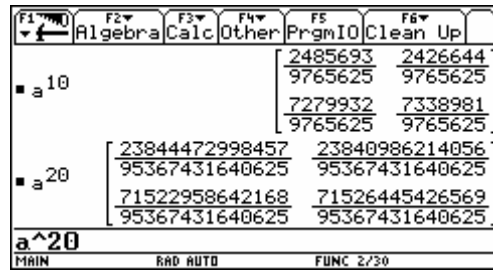
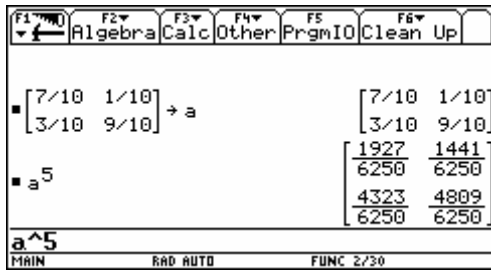
F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
a^{25}					
a^{30}					
$\begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix} \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix}$					

The limiting matrix appears to be of the form $L = \lim_{k \rightarrow \infty} A^k = \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix}$.

$$\begin{aligned} \text{The population stabilizes to } \lim_{k \rightarrow \infty} \left(A^k \cdot \begin{bmatrix} x_{2000} \\ y_{2000} \end{bmatrix} \right) &= \left(\lim_{k \rightarrow \infty} A^k \right) \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix} \\ &= \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix} \cdot \begin{bmatrix} 600000 \\ 4000000 \end{bmatrix} = \begin{bmatrix} 250000 \\ 750000 \end{bmatrix} \end{aligned}$$

The population distribution will eventually stabilize to the situation where 1/4 of the population is in the East and 3/4 of the population is in the West.

Note: When working in exact mode with fractions, the powers of the transition matrix A are correct but not useful:



Extension of question 2 - Category C4

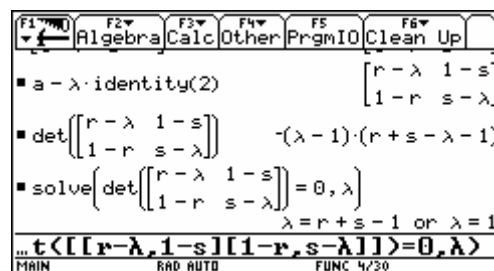
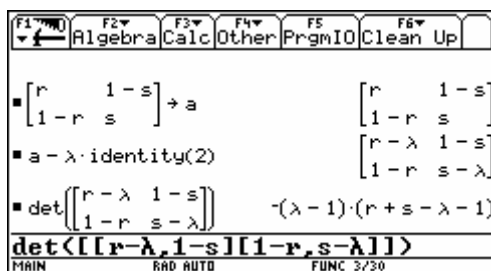
This extension was not a question on the entrance examination.

- d) Consider the transition matrix $A = \begin{bmatrix} r & 1-s \\ 1-r & s \end{bmatrix}$ ($0 \leq r \leq 1$ and $0 \leq s \leq 1$).
- Find the eigenvalues of A .
 - Find two linearly independent eigenvectors X_1, X_2 of A with corresponding eigenvalues λ_1, λ_2 .
 - Form the matrix P having X_1, X_2 as its column vectors and check that $D = P^{-1}AP$ is a diagonal matrix A with λ_1, λ_2 as its successive diagonal entries (P diagonalizes A).
 - Prove that $A^k = PD^kP^{-1}$.
 - Find $\lim_{k \rightarrow \infty} A^k$ and apply the result to the transition matrix of the population migration.

Solution

- (i) $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ is an eigenvector of A with eigenvalue λ if $AX = \lambda X$ or $(A - \lambda I)X = 0$.

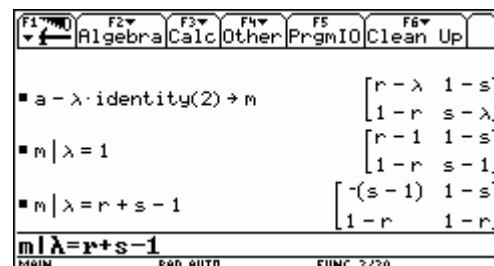
This homogeneous system of linear equations has a solution $X \neq 0$ if and only if $\det(A - \lambda I) = 0$.



The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = r + s - 1$.

- (ii) Case 1: $\lambda_1 \neq \lambda_2$ or $r + s - 1 \neq 1$
or $r + s \neq 2$

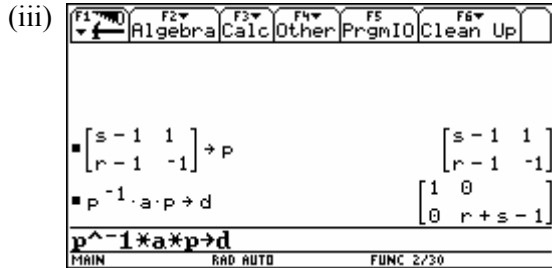
For $\lambda_1 = 1$, choose $X_1 = \begin{bmatrix} s-1 \\ r-1 \end{bmatrix}$



For $\lambda_2 = r + s - 1$, choose $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Case 2: $\lambda_1 = \lambda_2$ or $r + s = 2$ or $r = 1$ and $s = 1$

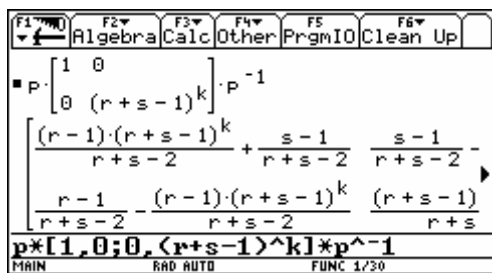
Then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



(iv) If $D = P^{-1}AP$ then $PD = AP$ and $A = PDP^{-1}$.

$$\begin{aligned} \text{Thus } A^k &= (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \quad (k \text{ factors}). \\ &= PD(P^{-1}P)D(P^{-1}P)\dots(P^{-1}P)DP^{-1} \\ &= PD^k P^{-1} \end{aligned}$$

$$\begin{aligned} \text{(v) } A^k &= PD^k P^{-1} = \begin{bmatrix} s-1 & 1 \\ r-1 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (r+s-1)^k \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ r-1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{(r-1)(r+s-1)^k}{r+s-2} + \frac{s-1}{r+s-2} & \frac{s-1}{r+s-2} - \frac{(r+s-1)^k(s-1)}{r+s-2} \\ \frac{r-1}{r+s-2} - \frac{(r-1)(r+s-1)^k}{r+s-2} & \frac{(r+s-1)^k(s-1)}{r+s-2} + \frac{r-1}{r+s-2} \end{bmatrix} \end{aligned}$$



As $0 \leq r \leq 1$ and $0 \leq s \leq 1$, it follows that $-1 \leq r + s - 1 \leq 1$

For $-1 < r + s - 1 < 1$, $\lim_{k \rightarrow \infty} (r + s - 1)^k = 0$; thus $\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} \frac{s-1}{r+s-2} & \frac{s-1}{r+s-2} \\ \frac{r-1}{r+s-2} & \frac{r-1}{r+s-2} \end{bmatrix}$.

For $r + s - 1 = 1$ or $r + s = 2$, $\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (see (ii), case 2).

For $r+s-1=-1$ or $r+s=0$ or $r=0$ and $s=0$, $\lim_{k \rightarrow \infty} A^k = \lim_{k \rightarrow \infty} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^k$ doesn't exist.

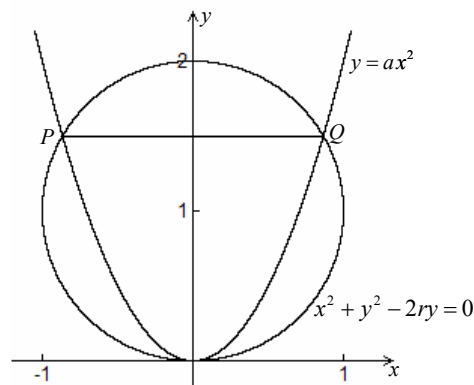
For the population migration matrix $A = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$ ($r=0.7$, $s=0.9$) the limit becomes

$$\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} \frac{s-1}{r+s-2} & \frac{s-1}{r+s-2} \\ \frac{r-1}{r+s-2} & \frac{r-1}{r+s-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

Question 3 - Category C4

Consider the circle with equation $x^2 + y^2 - 2ry = 0$ and the parabola with equation $y = ax^2$, where r and a are positive parameters with $r \cdot a > 1/2$.

- The parabola and the circle intersect at the origin and at the points P , Q . Find the coordinates of P and Q .
- Find the area A of the region above the parabola and below the line PQ .
- Which value of a (with constant r) gives the maximum area A ? Determine this largest area A_{\max} as a function of r .



Solution

a) $P = \left(\frac{-\sqrt{2ar-1}}{a}, \frac{2ar-1}{a} \right)$ and $Q = \left(\frac{\sqrt{2ar-1}}{a}, \frac{2ar-1}{a} \right)$.

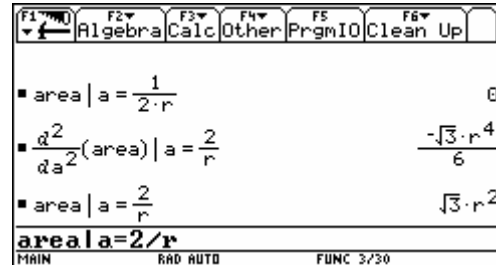
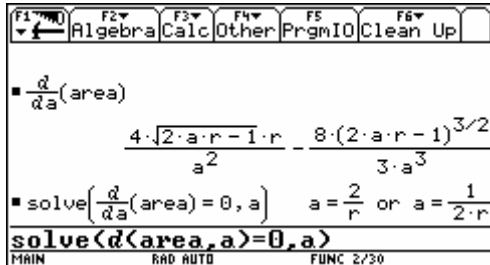
b) The area is $A = \frac{4(2ar-1)^{3/2}}{3a^2}$.

$-2 \cdot r \cdot y + x^2 + y^2 = 0 \mid y = a \cdot x^2$
 $-2 \cdot a \cdot r \cdot x^2 + a^2 \cdot x^4 + x^2 = 0$
 $\text{solve}(-2 \cdot a \cdot r \cdot x^2 + a^2 \cdot x^4 + x^2 = 0, x)$
 $x = \frac{-\sqrt{2 \cdot a \cdot r - 1}}{a} \text{ and } \frac{\sqrt{2 \cdot a \cdot r - 1}}{a^2} \geq 0 \text{ or } x = \rightarrow$
 $\text{solve}(\text{ans}(1), x)$

$2 \cdot \int_0^{\frac{\sqrt{2 \cdot a \cdot r - 1}}{a}} \left(\frac{\sqrt{2 \cdot a \cdot r - 1}}{a} - a \cdot x^2 \right) dx \rightarrow \text{area}$
 $\frac{4 \cdot (2 \cdot a \cdot r - 1)^{3/2}}{3 \cdot a^2}$
 $\text{... } a \cdot x^2, x, 0, \sqrt{2 \cdot a \cdot r - 1} / a \rightarrow \text{area}$

c) An extremum for A can occur when $\frac{dA}{da} = 0$ or $a = \frac{2}{r}$ (interpret the second solution $a = \frac{1}{2r}$).

For geometrical reasons we expect a maximum and this is confirmed by the negative sign of the second derivative for $a = \frac{2}{r}$. The maximum area is $A_{\max} = \sqrt{3}r^2$.



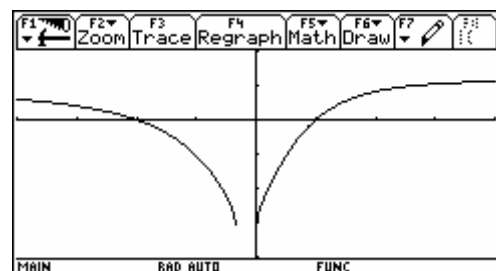
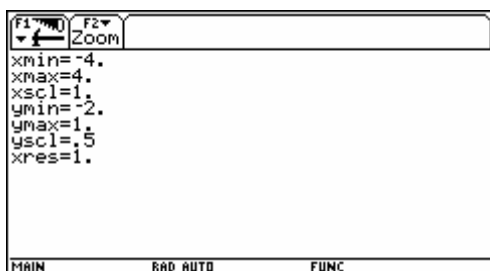
Note: CAS can do the correct calculations, but the student must know that:

- the coordinates of P and Q can be found by solving a system of equations,
- the area of a region between two graphs can be expressed as an integral,
- an extremum of a differentiable function occurs at a point where the derivative equals zero and changes sign.

Question 4 - Category C0

The figure shown gives the graph of a real function f , defined as follows $f(x) = \arcsin\left(\frac{x^2 + ax + b}{cx^2 + d}\right)$.

with $a, b, c, d \in \mathbb{R}$. The graph has a horizontal asymptote with equation $y = \frac{\pi}{6}$, intersects the x -axis at the points $(1, 0)$ and $(-2, 0)$ and intersects the y -axis at $\left(0, -\frac{\pi}{2}\right)$.



- Find a, b, c and d .
- Are there other asymptotes? If yes, find the equation(s).
- Determine the domain D of f .

Solution

a) horizontal asymptote $y = \frac{\pi}{6}$: $\lim_{x \rightarrow \infty} \arcsin\left(\frac{x^2 + ax + b}{cx^2 + d}\right) = \frac{\pi}{6}$ or $\arcsin\left(\frac{1}{c}\right) = \frac{\pi}{6}$ or $c = 2$.

$$(1, 0) \in f : \quad \arcsin\left(\frac{1+a+b}{2+d}\right) = 0 \quad \text{or} \quad 1+a+b=0$$

$$(-2, 0) \in f : \quad \arcsin\left(\frac{4-2a+b}{8+d}\right) = 0 \quad \text{or} \quad 4-2a+b=0$$

$$\left(0, -\frac{\pi}{2}\right) \in f : \quad \arcsin\left(\frac{b}{d}\right) = -\frac{\pi}{2} \quad \text{or} \quad \frac{b}{d} = -1$$

It is easy to find the solution $a = 1, b = -2, c = 2, d = 2$. As a check, the graph can be drawn.

b) The function $f(x) = \arcsin\left(\frac{x^2 + x - 2}{2x^2 + 2}\right)$ has no other asymptotes.

There is no oblique asymptote because the horizontal asymptote $y = \frac{\pi}{6}$ is for $x \rightarrow \pm\infty$.

A vertical asymptote is impossible because the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of the arcsin-function is bounded.

c) The domain D of the function is the set of points x for which

$$\begin{aligned} -1 &\leq \frac{x^2 + x - 2}{2x^2 + 2} \leq 1 \\ &\Downarrow \\ -2x^2 - 2 &\leq x^2 + x - 2 \leq 2x^2 + 2 \quad (2x^2 + 2 \text{ is positive!}) \\ &\Downarrow \\ 3x^2 + x &\geq 0 \quad \text{and} \quad x^2 - x + 4 \geq 0 \\ &\Downarrow \\ x(3x + 1) &\geq 0 \quad \text{and} \quad x \in \mathbb{R} \\ &\Downarrow \\ x &\leq -\frac{1}{3} \quad \text{or} \quad x \geq 0 \end{aligned}$$

We conclude that $D = \left]-\infty, -\frac{1}{3}\right] \cup [0, \infty[$.

Remark: *This example illustrates that CAS is not always of practical use. CAS is only one of the many tools in mathematics and one should use the right tool at the right moment.*

6.5 Final exam in Victoria, Australia

In the context of a pilot project, the Victorian Curriculum and Assessment Authority developed a different examination which allowed CAS. This was in addition to the existing examinations which assumed access to graphing calculator, which continued to be developed in their own right for that accredited study. These exams are for Year 12, which is the final year of secondary school. The Mathematical Methods(CAS) were the first 'CAS-assumed' exams in this state of Australia. The students did two exams (one was a multiple choice and short answer exam and the second one was the analysis task). 75 students were involved in a pilot study where students were allowed to use CAS in their examinations.

In the following a comparison is made between the Non-CAS and the CAS-Assessment (Written Examination 2 – Analysis task). In 2003 more schools will be allowed to offer the new subject where CAS can be used in the examinations and the project group members are very interested to see how these CAS-exams evolve over the next couple of years

The CAS examinations (sample papers, supplementary questions, solutions, comments and advice in 2001, and actual examinations in 2003) were newly developed for the pilot study, and contain both common material and distinctive material with respect to the usual Mathematical Methods examinations.

The VCAA website, www.vcaa.vic.edu.au/vce/studies/MATHS/caspilot.htm, contains a comprehensive dedicated section for the CAS pilot which includes sample teaching and learning tasks, discussion papers and the like as well as links to the sample examinations, 2002 *examinations* www.vcaa.vic.edu.au/vce/Exams/Maths/MathsMethodsCAS.htm, (and soon the 2003 examination): and the examiners reports for the 2002 examinations, www.vcaa.vic.edu.au/VCE/Assessmnet/AssessReports/Maths/MathsMethodCAS.htm.

More information together with a rich source of papers on the use of CAS in Mathematics Education can be found at www.edfac.unimelb.edu.au/DSME/CAS-CAT.

Many thanks to Lynda Ball, University of Melbourne, for her valuable help and support.

Victorian Certificate of Education 2002

Written Examination 2 (Analysis task)

See references at the end of the paper

An Assessment for two groups of students

Reading time 15 minutes

Working time 90 minutes

Non CAS Assessment

Students are permitted to bring
.... and an approved scientific and/or
graphics calculator (memory may be retained).

Instructions

- Answer **all** questions in the space provided.
- A decimal approximation will be not accepted if an exact answer is required to a question.
- Where an exact answer is required to a question, appropriate working must be shown.
- Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Total 55 marks

CAS-Assessment

Students are permitted to bring
.... and an approved scientific and/or
computer algebra (CAS) calculator (memory
may be retained).

Instructions

- Answer **all** questions in the space provided.
- A decimal approximation will be not accepted if an exact answer is required to a question.
- Appropriate work must be shown if more than one mark is available.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Total 55 marks

Question 1

A well-designed computer screen display aims to make it quick and easy for a user to do tasks such as clicking on a screen button. Fitt's Law models the way in which the time taken to move to, and click on, a screen button depends on the distance the mouse is moved and the width of the screen button. According to Fitt's Law, for a fixed distance travelled by the mouse, the time taken, in seconds, is given by $a - b \log_e(x)$, $0 < x \leq 5$, where x cm is the button width and a and b are positive constants for a particular user.

a) Minnie discovers that, for her, $a = 1.1$ and $b = 0.5$

(i) Let $f: (0,5] \rightarrow \mathbb{R}$, $f(x) = 1.1 - 0.5 \log_e(x)$.

Sketch the graph of $y = f(x)$ on the axes below. Label any asymptote with its equation and any end-point with its exact coordinates.

3 marks

(ii) Explain why f^{-1} , the inverse function of f , exists.

1 mark

(iii) Find $f^{-1}(x)$, the rule for f^{-1} .

2 marks

(iv) State the domain of f^{-1} .

1 mark

(v) Sketch the graph of $f^{-1}(x)$ on the axes below. Label any asymptote with its equation and any end-point with its exact coordinates.

2 marks

- b) Mickey decides to find values of a and b for his use. He finds that when $x = 1$, his time is 0.5 seconds, and when x is 1.5, his time is 0.3 seconds.

Non-CAS-Assessment

- c) Show that, when the button width is halved, the time taken by Minnie (for whom $a = 1.1$ and $b = 0.5$) is increased by $\log_e \sqrt{2}$ seconds.

3 marks

CAS-Assessment

- c) Solve the equation $k(1.1 - 0.5 \log_e(x)) = T$ for x , where k and T are positive real numbers.

1 mark

- c) Let $g: [0,5] \rightarrow R$, $g(x) = k(1.1 - 0.5 \log_e(x))$, where k is positive and real. Given a positive real number T , find the largest value of k such that the equation $g(x) = T$, has a solution for x in the domain of g .

2 marks

Question 1

Total 14 marks

Question 1

Total 14 marks

Question 2

Emmy is gathering data on two particular species of yellow butterflies, Fhaisi and Jojo, which are very difficult to tell apart. Both species are equally likely to be caught.

One technique for telling the difference between the two species is by measuring the length of their antennas. For Fhaisi butterflies, antenna lengths are normally distributed with a mean of 20 mm and a standard deviation of 2 mm.

- a) Find the probability, correct to three decimal places, that a randomly selected Fhaisi butterfly antenna is shorter than 16 mm. 2 marks
- b) 8% of Jojo butterfly antennas are shorter than 19 mm and 8% of Jojo butterfly antennas are longer than 28 mm. Assume that the antenna length of Jojo butterfly antennas are also normally distributed. Find the mean and the standard deviation of antenna length of Jojo butterflies, to the nearest 0.1 mm. 4 marks
- c) In the region where Emmy is butterfly hunting, 20% of the yellow butterflies are Jojos and the other 80% are Fhaisis. The probability that a randomly selected Jojo butterfly antenna is shorter than 20 mm is 0.1370.
- (i) Calculate the proportion, correct to three decimal places, of **all** Jojo and Fhaisi butterfly antennas that are shorter than 20 mm. 2 marks
- (ii) Emmy examined a single butterfly antenna. It was shorter than 20 mm. What is the probability, correct to three decimal places, that it is a Fhaisi antenna? 2 marks

Non-CAS-Assessment

- (iii) Find the probability, correct to three decimal places, that a random sample of 10 yellow butterflies from this region will contain exactly 4 Jojo butterflies.

2 marks

Question 2

Total 12 marks

Non-CAS-Assessment

Question 3

- a) Write down an equation in x , the solutions of which give the x -coordinates of the stationary points of the curve whose equation is

$$y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x).$$

2 marks

- b) The diagram shows the curve whose equation is

$$y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$$
 and the normal to the curve at A , where $x = 1$.

- (i) Show that the equation of this normal is $y = x - 1.5$.

3 marks

CAS-Assessment

- d) Let X be the random variable with values equal to the distance in metres of a Fhaisi butterfly from an old tree.

The probability density function of X is

$$f(X) = \begin{cases} \frac{2x}{a^2} & 0 \leq x \leq a, \quad a \text{ is a constant} \\ 0 & \text{otherwise} \end{cases}$$

- (i) It is found that the mean distance of a butterfly from the old tree is 150m. Show that the value of a is 225.

2 marks

- (ii) Find the probability, correct to three decimal places, of a Fhaisi butterfly being within 200 m of the old tree.

2 marks

Question 2

Total 14 marks

CAS-Assessment

Question 3

- a) (i) The polynomial $2x^4 - x^3 - 5x^2 + 3$ can be factorised as $x(2x - 3)(ax^2 + bx + c)$.

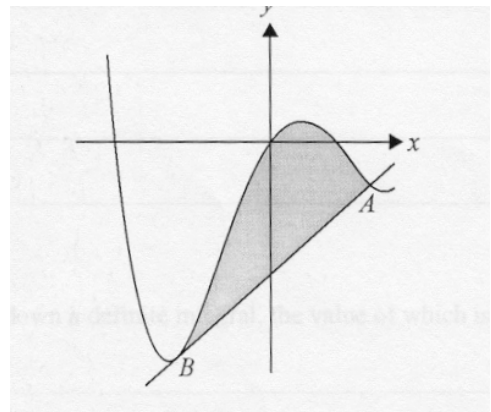
Find the values of a , b and c .

1 mark

- (ii) Find the exact value solutions of the equation

$$2x^4 - x^3 - 5x^2 + 3x = 0.$$

2 marks



- (ii) Show that this normal is a tangent to the curve at a point B and find the exact values of the coordinates of B . *4 marks*

- c) (i) Write down a definite integral, the value of which is the area of the shaded region. *2 marks*

- (ii) Find the area of the shaded region, correct to two decimal places. *1 mark*

Non-CAS-Assessment

CAS-Assessment

Question 3

Total 12 marks

Question 3

Total 13 marks

Question 4

On an adventure park ride, riders are strapped into seats on a platform which starts 15 metres above the ground and goes up and down. The distance, x metres, of the platform above the ground, t seconds

after the ride starts can be modelled by the formula $x(t) = 15 + 6 \sin\left(\frac{\pi t}{3}\right)$.

- a) (i) According to this model, find the maximum height above the ground reached by the platform.

1 mark

Non-CAS-Assessment

CAS-Assessment

- (ii) According to this model, how many seconds after the ride starts is the platform first closest to the ground and how high above the ground is it at that time?

2 marks

- (ii) According to this model, how many seconds after the ride starts is the platform first exactly 9 metres from the ground?

1 mark

Tasmania Jones is redesigning the ride so that the platform moves further up and down each cycle. During the first 60 seconds of the redesigned ride, the distance, y metres, of the platform above the ground, t seconds after the ride starts, can be modelled by the formula

$$y(t) = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right), \quad 0 \leq t \leq 60.$$

- b) (i) According to this model the platform is exactly 6 metres above the ground for the first time about 58 seconds into the ride. Find this time correct to two decimal places of a second.

1 mark

- (ii) According to this model, how many times is the platform exactly 15 metres above the ground from $t = 40$ to $t = 59$? *1 mark*

- (iii) According to this model, find the time which passes from when the ride starts until the platform first reaches 24 metres above the ground. Give your answer correct to the nearest second. *2 marks*

c) (i) Find an expression for $\frac{dy}{dt}$. 2 marks

(ii) Hence write down an equation, one solution of which is the value of t , when the platform is closest to the ground. Find this value of t , correct to two decimal places. Also find, according to the model, the distance of the platform above the ground at that time, correct to two decimal places of a metre. 3 marks

d) Tasmania can adjust the ride so the model for the distance, in metres, of the platform above the ground t seconds after the ride starts becomes

$$h(t) = 15 + ae^{0.04t} \sin\left(\frac{\pi t}{3}\right), \quad 0 \leq t \leq 60, \text{ where } a \text{ is a positive constant.}$$

Find, correct to three decimal places, the greatest value of a such that h' is never more than 11 during the first 60 seconds of the ride.

3 marks

Non-CAS-Assessment

CAS-Assessment

e) Safety regulations for the ride require that

$$-11 \leq \frac{dy}{dt} \leq 11 \text{ during the first 60 seconds.}$$

Find the range of values of t , correct to three

decimal places, for which $-11 \leq \frac{dy}{dt} \leq 11$

during the first 60 seconds of the ride.

2 marks

Question 4

Total 17 marks

Question 4

Total 14 marks

References

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Victorian Curriculum and Assessment Authority (2002). *VCE Mathematical Methods (CAS) Pilot study: Written examination 1*. Melbourne: Author.

7. Argumentative discussion – ADVOCATUS DIABOLI

Having now considered and summarised the available research, examined various teaching approaches, and how assessment might be carried out, this final section asks and responds to some searching questions regarding the use of CAS.

- a) Mathematics educators in the 80's and 90's misjudged the central role of maintaining a balance between basic skills and use of the calculator. Is there not a danger that we may repeat with CAS the errors we made in taking for granted the use of four function calculators – but on another level?

The mistakes made with four function calculators have been widely recognised and considerable efforts have been made in order to avoid this over-reliance. The worldwide programme of training, via –T3 for example, ensures that teachers have access to clear guidance as to how best to use the technology and how to integrate it into teaching and learning to ensure it is used in an optimum manner. The extensive research and pilot projects, such as the Austrian ACDCA, Australian CAS-CAT, and the various CAS conferences, help to ensure that classroom practice is well informed. Many Universities now include the use of CAS as part of the initial teacher training programme to prepare new teachers for the challenges which use of CAS presents. Finally, many countries have PAP (Pencil and Paper) tests to ensure that the by-hand skills are kept under review and to the fore.

- b) Studies show that students do not lose their basic 'by hand skills' when using CAS but I am less than convinced of this. What evidence is available which proves they do not become over reliant on CAS to the detriment of these skills?

As indicated earlier in the text of this book the research evidence is very clear. Use of CAS does not appear to adversely affect 'by hand skills'. Conversely, evidence points to an improvement of these skills. Students require to be more flexible and need to interpret different algebraic representations when using CAS, often dealing with unexpected outcomes. The issue is very much in the public domain and most teachers split time – doing things without machine as well as with CAS. The Australian CAS-CAT project and the debate started by Kutzler et al, to name but two, help ensure that the issue of what must be done by hand is kept under review. CAS in hand held form allows instant access to CAS but most importantly teachers and students are aware that there are times when they must SWITCH OFF!

- c) Enthusiasts tend to be optimistic and identify the benefits and possibilities of using new technology, but what are the dangers or drawbacks of using CAS technology?

Enthusiasts generally want to impress, want to show all that CAS can do. They need however to consider what is relevant (less is more?) They also need to be aware that what is natural and easy for them is not necessarily natural or easy for many teachers. (Monaghan, 2002) An overall awareness is required of these factors and to avoid old problems being made too (artificially) complex to justify use of CAS. Traditional problems can remain – polynomial of degree 3 can now handle one of degree 4 etc. However, if CAS is not used in a suitable way (see teaching sequences?), its use may serve only to widen the gap between gifted and ungifted pupils.

- d) As mentioned above, the gap between the weaker student and the more able is already large, will using CAS not widen this gap, since there is more emphasis on Formulation and higher order strategies and less on process skills? Furthermore, is there not a danger that a subject already perceived by students as ‘difficult’ will be made to seem more difficult and thus discourage students from studying further?

Weak students are those who also have difficulty with process skills. However, before the use of CAS the same students must have been frustrated because many of them didn’t like training of skills. Rearrangement of terms for example, often prior to the need, made no sense to them. On the other hand, use of CAS provides many opportunities for weaker students to show their creative side, to involve themselves in discussion about the mathematics, and to find more ways to express themselves.

With CAS more ways to solving problems are made possible. ‘Social learning’ with problems takes place, helping each other with hardware and software, exchanging devices comparing answers, cooperation and brighter helping weaker – the result being an overall change of ‘mood’ in class! Weaker students often have good ideas but were ‘blocked’ by manipulation. CAS now sets them free to explore these, and being no longer frustrated – enjoy maths! Using CAS allows some content to be taught at an earlier stage is a well used argument and can be used in either the black box and white box approaches as required.

One of the authors of this book remarked when asked these questions:

When using CAS you become aware that most of the exercises in our textbooks are prepared to get “good” results in order to be solvable by-hand. With CAS you don’t have these restrictions. You can e.g. introduce parameters, use realistic data etc., see the result of changes in parameters and data immediately. This often leads to interesting discussions of mathematical models depending on these parameters. Weaker students are mostly those who have also difficulties with or no interest in process skills. In my CAS classes the gap between weaker and more able students was significantly smaller and I had fewer students who dropped studies. With the time you earn with less process skills, you can treat with more interesting demanding tasks which raises the mathematical understanding and interest of the students.

- e) `Where is the evidence to show that students ARE more motivated when using CAS (as opposed to graphing calculators)? Is this only anecdotal or is there clear research evidence to support this?

The evidence for this is well supported in the research section of this document and full analysis of ACDCA reports 1 to 4 and personal evidence from the authors concurs. Perhaps the students should have the last word here as indicated in their answers to a questionnaire.

How much did/do you like Mathematics

- (i) Before working with the TI-92:

	1	2	3	4	5	
did not like	153	352	495	287	114	liked very much

- (ii) Since working with the TI-92:

	1	2	3	4	5	
do not like	111	218	436	466	171	like very much

Evaluation by “Centre for School development” ACDCA – Report

f) What can be done with CAS that cannot be done with a graphing calculator?

There are many answers to this as can be seen in the teaching examples. When using CAS the authors believe that students can: generalise with CAS after exploration and investigating, define functions with more than 1 variable, perform the symbolic part of working with functions which cannot be done with the TI-83 Plus for example. They can see multiple representations of a problem which now includes algebraic as well as tabular, graphical, and numerical ones. Teachers can treat examples differently (earlier) and other didactical approaches become possible. Some interesting subject areas become more accessible because CAS takes care of time consuming work and weaker students can obtain exact results from calculations. Students are able to check results at various stages before progressing on to the next part of the problem. CAS can be used to improve basic manipulation skills – algebraic equivalence and fractions in particular – pattern recognising a strength. CAS sets students ‘free’ to choose own values – frees them from drudgery- but with the responsibility to interpret results! Tables of formulae are no longer necessary.

Again one of the authors commented:

Solving a demanding task with parameters and CAS needs as much skill as 10 routine tasks without CAS. Routine tasks are often made without reflection. This means you can get more with fewer demanding tasks with reduced skills than with a lot of routine tasks. I compared exercises by-hand with CAS classes which I made with former classes without CAS in the domain of derivations. Astonishing results: CAS classes were better although they had much less training with the subject. By-hand you have to follow a strict order in teaching. The scaffolding method with CAS allows you to treat subjects which are much later in the curriculum or even not possible by-hand.

8. Concluding remarks

In The CASE for CAS the authors set out to show ‘beyond reasonable doubt’ that teaching with CAS was ‘innocent’ of eroding the by-hand skills of mathematics students. We feel that we have gone beyond that aim and provided a definitive CASE for the use of CAS by all students of mathematics.

In summing up, provided students have access to CAS technology at home and in the classroom, effective use of CAS can have a number of important benefits. These include:

- Making concepts easier to teach because students can approach situations numerically, graphically and symbolically,
- Supporting visualisation and allowing situations to be explored that cannot otherwise be, thus enabling students to take mathematics to a more advanced level,
- Saving time on routine calculations – time that can be exploited to study effects, outcomes and ‘what if...?’ Situations,
- Improving the students’ perception of mathematics and consequently increasing their enjoyment of the subject and their motivation to learn.

We live in a mathematical world. The level of mathematical thinking and problem solving needed in the workplace has increased dramatically. In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes these doors. But the vision of mathematics teaching and learning is not the reality in the majority of classrooms and schools. In this age of Information Technology students need to learn a new set of mathematics basics that enables them to solve problems creatively and resourcefully. (Heugl, 2000)

The authors of this book believe that teaching and learning with CAS can transform these classrooms. We hope you will make the above vision a reality and find in favour of The CASE for CAS.

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10. About the authors

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was a teacher in a Business Administration College (Handelsakademie) in St. Pölten, Lower Austria for more than 30 years. Here he started very early teaching mathematics supported by technology. During his career as teacher he gave numerous courses on this issue. Since 1999 he has been T³ coordinator for Austria and is still busy as an active member of the ACDC (Austrian Center of Didactics of Computer Algebra). In his leisure time he enjoys being grandfather of six grandchildren and likes to play tennis, go for extended hikes and to travel around the world to attend mathematics conferences.

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René Hugelshofer

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Gert Schomacker

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