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Realizing the concept of "Multiple Representations" by using CAS

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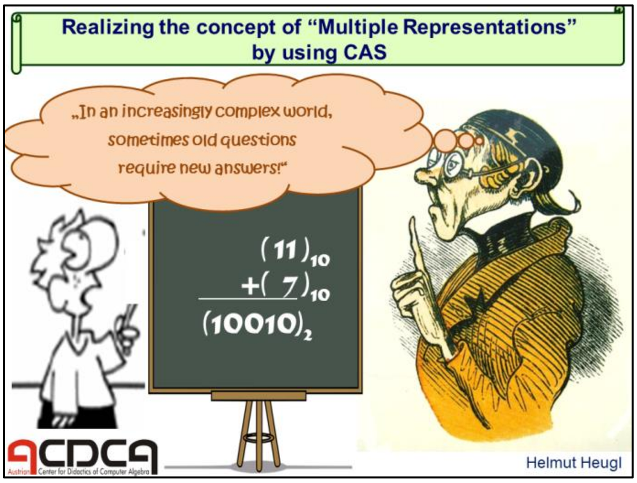
Mathematical concepts are presented in multiple modes of representation (or "prototypes") such as text, graphs and diagrams, tables, algebraic expressions and computer simulations. A prime goal of teaching is to help learners develop an understanding of the mathematical concepts by considering and using these different representational modes and levels. Several prototypes of a concept provide complementary information [1].

Therefore it is not enough to become acquainted with and to understand the information of a certain representation mode. A central cognitive activity on the way to mathematical concepts is to build links between representation modes of a concept. In traditional mathematics education prototypes mostly are available in a serial way. The main importance of technology tools is that the learner can use several prototypes parallelly.

By using examples of Algebra and Analysis I will show the role of CAS when building links between several representation modes of a concept or when solving problems [2].



"***Old questions require new answers***"!

2004 I have given a lecture here in Montreal and I I have spoken about of the results of our CAS-research projects.

One result we have called [Heugl, 1996]
"**The Window-Shuttle-Method**"

Meanwhile we can find many investigations and new answers of this important didactical principle, but the new answers also have a new heading:

"**The principle of multiple representations**"

The potential for computer-based aids to learning mathematics remains high, although the current contribution of technology to didactic and content innovations is at least in our country frustratingly low. Discussions about the role of technology are too often based on what computers can do rather than on research-based investigations of how students learn with technology. In this lecture I will discuss the question. "How do students learn with modern digital media which can be characterized as „Multi-Representations-Programs" (MRP) and what is the role of CAS among the offered modes of representations.

1. Mathematical prerequisites

Before describing the principle of "Multiple Representations" and showing examples concerning the role of CAS some mathematical prerequisites are necessary.

We cannot discuss the question "Why CAS?" without asking "Why mathematics?" Diverse answers can be found in the mathematical literature and also in every mathematics curriculum of every country all over the world. I prefer the answer of my important mentor, a famous Austrian mathematician, Bruno Buchberger:

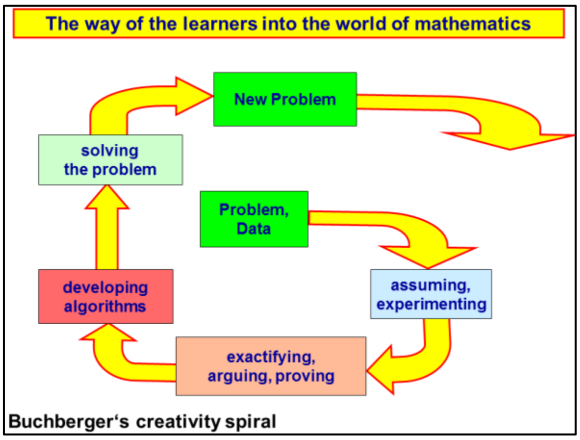
"mathematical thinking technology is the essence of science and the essence of a technology based society"

Interpreting a mathematics curriculum we should not require: "*our students have to learn integrals!"*, we should question: *„what thinking technology students are gaining when learning integrals?"* This definition answers not only the question "Why learning mathematics?", it influences also the question: "How learning mathematics?" and therefore it plays an important role in the topic of my lecture.

The way of the learners into the world of mathematics:

For describing the learner's way into the world of mathematics Bruno Buchberger uses a spiral as a model [Buchberger, 2002].

The spiral begins with observations, data material or **problems**, the solution of which can generate the development of algorithms or in the creation of new concepts.

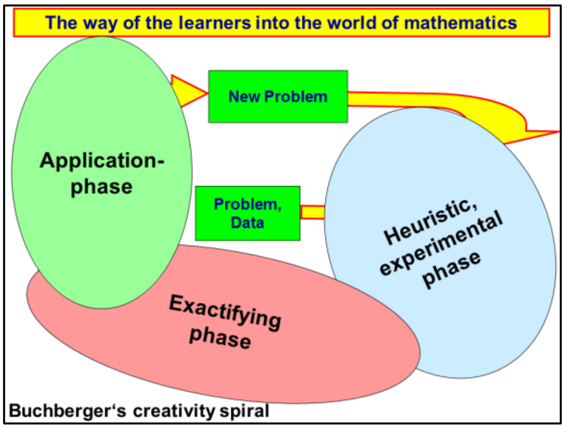


By analyzing, **experimenting** or generally through heuristic strategies, assumptions are found for solving the given problem.
By proving and substantiating, in other words, by **exactifying,** one enters into the next stage of the spiral: Theorems and sentences which can now be ensured and proved.
Thus, supported by acquired knowledge, one proceeds **to develop** those **algorithms** or programs which are necessary for problem solving. Testing of and consolidating the developed algorithms by practicing is an integral part of this stage.
The actual part of the spiral ends with the next step. The insight and strategies are now used **to solve the initial problem** or related problems.

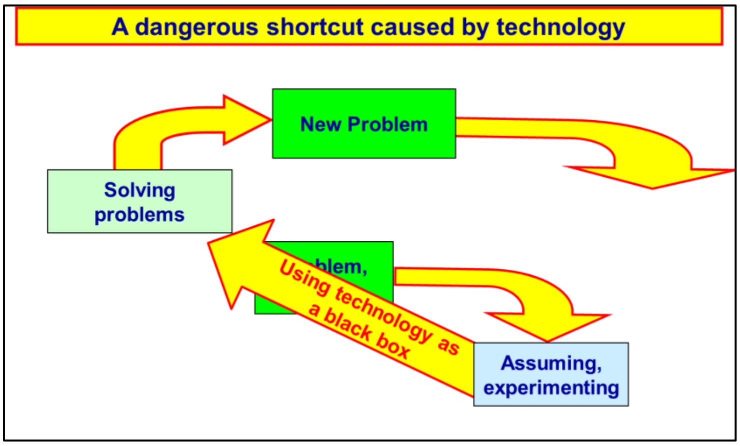
When **new problems** evolve and new additional knowledge is necessary or new algorithms need to be developed, then the stages of the spiral are repeated once again.

The several steps in course of one pass of the spiral can be seen as activities in **three phases**:

- The heuristic, experimental phase
- The exaxtifying phase
- The application phase



Observing mathematic classes where technology is used we find **a dangerous shortcut** of the creativity spiral:



The experimental phase does not replace the exactifying phase - it is the preparation !

The pupil could form assumptions in the heuristic phase, skip over the abstract phase, over the corroboration of algorithms and the practicing of calculating skills and then using the CAS as a Black Box immediately turn to the applications.

This way of doing mathematics does not establish the thinking technology that is demanded in our curriculum.

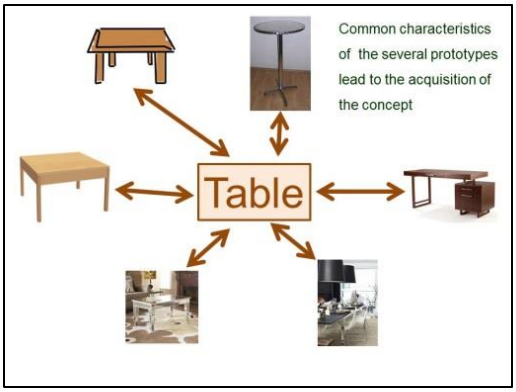
2. **The „principle of multiple representations“**

There is an important gap between mathematical concepts and concepts in other scientific fields. We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. With other words we need representations to develop them and to work with them.

2.1 Learning by exploring prototypes (representations)

General concepts become cognitively available through concrete representations or in our words through prototypes of the concept [Dörfler, 1991}

Think about the concept “table”. How does a child acquire this concept? Not by an exact definition given by mother or father. The child experiences several prototypes of tables and the common characteristics of the several prototypes lead to the acquisition of the concept.



Mathematical representations can be categorized in two main classes – symbols and icons

- **Descriptive representations** include information given by words or sentences in the natural language or symbols and objects in the mathematical language or data e.g. offered in tables
- **Depictive representation** provide visual information like graphs, diagrams, geometric objects, flow charts, a.s.o.

2.2 Learning by interacting between „prototypes“ (representations)

The availability of single prototypes is not enough for a successful learning process. The central cognitive activity is the permanent variation of the representations and the interaction between the several prototypes. The analysis of the use of representations in context of mathematics education shows that a close interaction between depictive and descriptive representations is needed order to make the best use of both kinds of representations for successful thinking and problem solving.

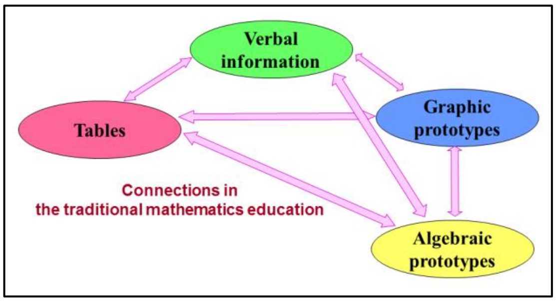
> ***We consider the combined use of different representations - especially descriptive and depictive ones – as a key concept for teaching mathematics and for thinking and problem solving in mathematics*** [Schnotz, 2010]

This quotation can be seen as a **definition of the "Principle of multiple representations"**

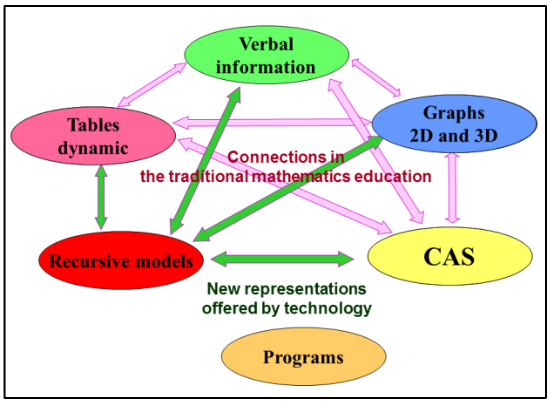
On the way of the learners into the world of mathematics the concept of functions play a central role.

Following prototypes (representations) of functions are used:

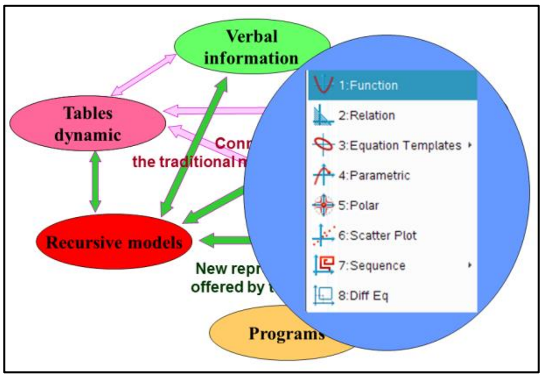
In traditional mathematics education following prototypes (representations) of functions are used, some descriptive and others depictive.



The computer as a medium of prototypes*:* *The computer changes the various prototypes and offers those which would not be available without the computer.*

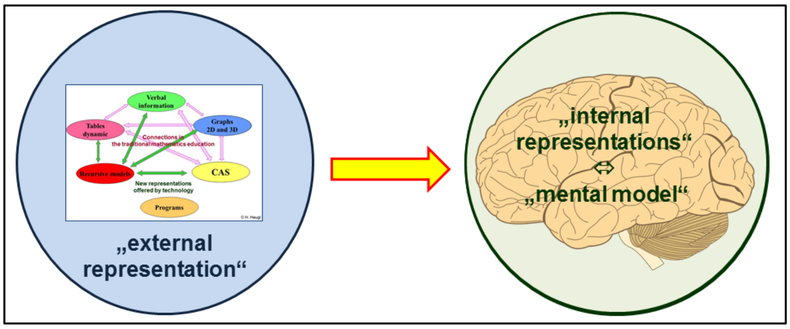


If we use a learning media like TI Nspire CAS, GeoGebra or Casio Classpad and look at the graphic prototype we can see a lot of graphic representations



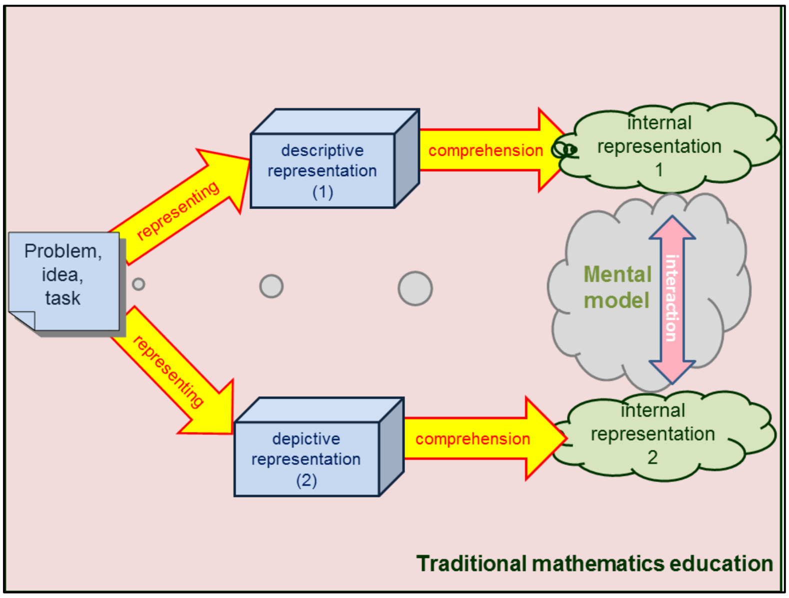
2.3 From external to internal representations

> *The sense and purpose of learning by using multiple representations is the construction of internal (mental) representations on the basis of intentionally created external representations. The distinction between descriptive and depictive representations applies also to the internal (mental) representations constructed during comprehension [Kintsch, 1998].*



I will try to visualize the difference between the development of a mental model on the one hand without using technology in traditional mathematics education and on the other hand when using technology:

Development of a mental model in traditional mathematics education

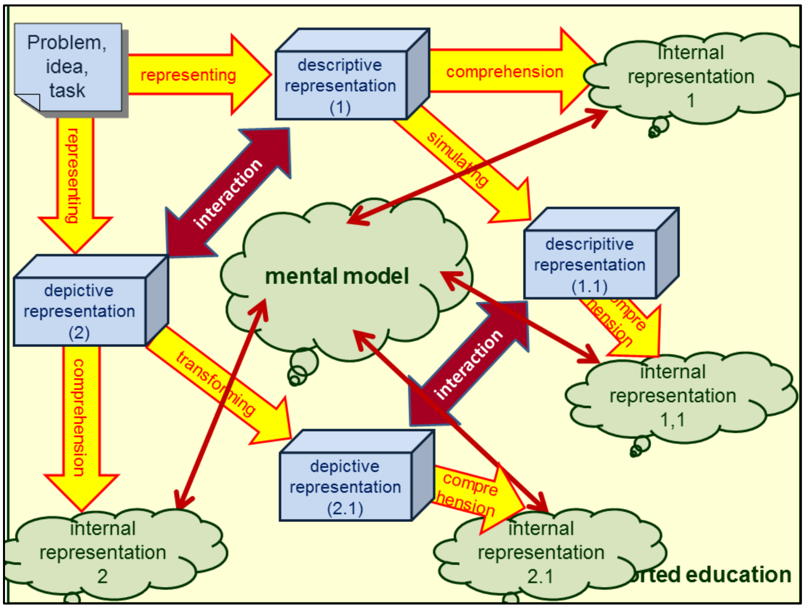


More frequently only one representation (either descriptive or depictive) is developed. Rarely the interaction between several representations can be observed.

One of the reasons is that representations can only be developed serially and therefore interactions between representations are more difficult.

Development of a mental model when using modern digital media

(⇔ “multi representation tools”)



(1) Depictive and descripitive representations can be developed parallelly and therefore interactions are much easier.

(2) Representations can be changed by transforming with technology or by simulating ⇨ new internal representations are developed

(3) The resulting mental model is based on several interacting internal representations.

Advantages of the use of technology:

- ☞ The development of representations is much easier
- ☞ Technology offers new representations e.g. recursive models , 3D-graphic, a.s.o.
- ☞ Transforming representations and especially simulation is at first possible by using technology
- ☞ Interaction between representations is easier and representations are available parallely
- ☞ Technology supported interaction between representations produces better mental models

Modern digital media are „Multi-Representations-Programs“ (MRP) ⇨

> **Modern digital media are a prerequisite for the realization of the principle of multiple representations.**

3. The role of CAS for realizing „the principle of multiple representations“

3.1 Interaction between verbal representations and symbolic representations of CAS

Symbolic representations are also elements of a language the mathematical language:

> ***Mathematics is a language*** *and like other languages it has its own grammar, syntax, vocabulary, word order, synonyms, conventions, a.s.o.*
>
> *This language is both a means of communication and an instrument of thought.*
>
> *[W. Esty, 1997]*

The development of this language over the centuries:

The ancient mathematical language was close or identical to the native language	**In the course of history more and more new language elements, grammar and syntax were developed**
“If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.” (Euclid, Elements, II.4, 300 B.C.)	The equivalent rule using algebraic language elements: $(a+b)^2 = a^2 + b^2 + 2.a.b$
“The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.” (Archimedes, On the Sphere and the Cylinder, 220B.C.)	$A = r.2\pi r/2 = r^2\pi$

These examples and our investigations in the classroom corroborate J. Piagets thesis that the genesis of knowledge in the sciences and in the individual follows the same mechanisms. Also students during their way into mathematics acquire more and more new language elements and necessary rules for using the language of mathematics for problem solving.

If we maintain that the main role of mathematics is problem solving, consisting of the activities modelling – operating – interpreting, then a main goal of mathematics learning is the translation process from a problem formulated in the native language to a mathematical model written in the language of mathematics.

Technology and especially CAS support the translation process from the native language into the language of mathematics.

Example 1: Derivation of Cramer's rule

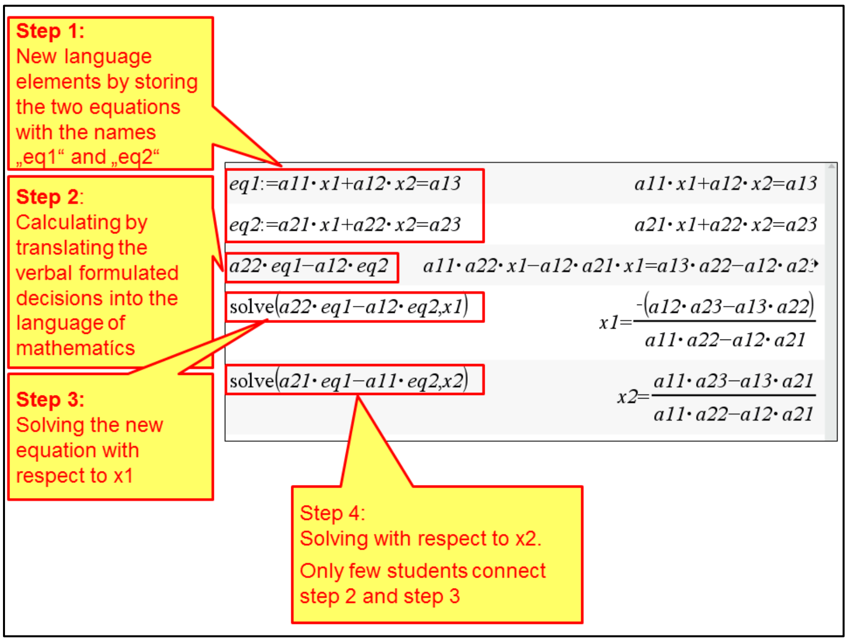
Given is a system of equations with two equations and two variables:

$$(1) \quad a11 \cdot x1 + a12 \cdot x2 = a13$$

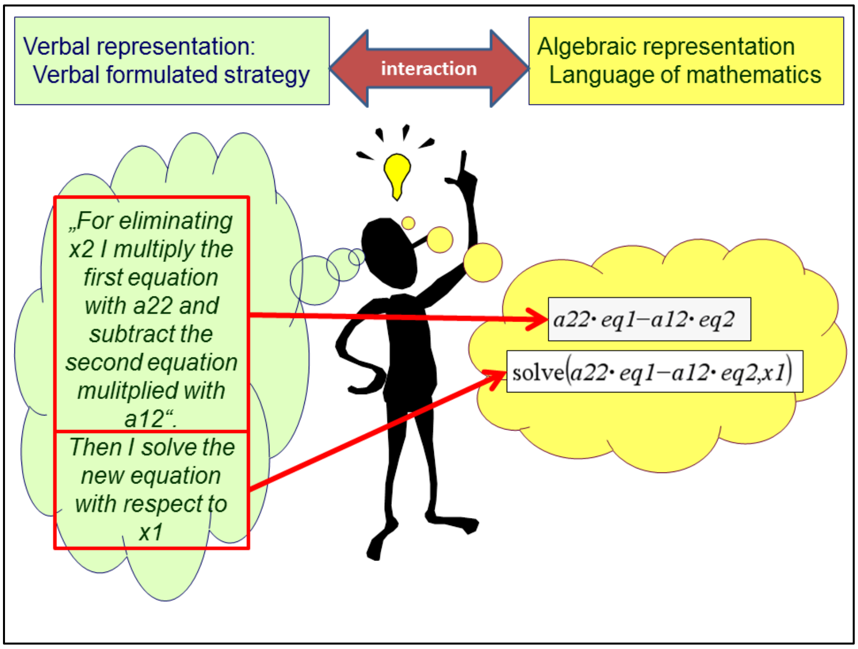
$$(2) \quad a21 \cdot x1 + a22 \cdot x2 = a23$$

Task:

Find a solution by using CAS



The advantages of the use of CAS:



The verbal formulated decisions of the verbal representation are directly translated into the language of mathematics

Example 2: Flu epidemic

For investigating the development of a flu wave in a town with 5000 inhabitants the number of sufferers E with respect to the time t (in days) can be described with a cubic polynomial function.

Following information are known:

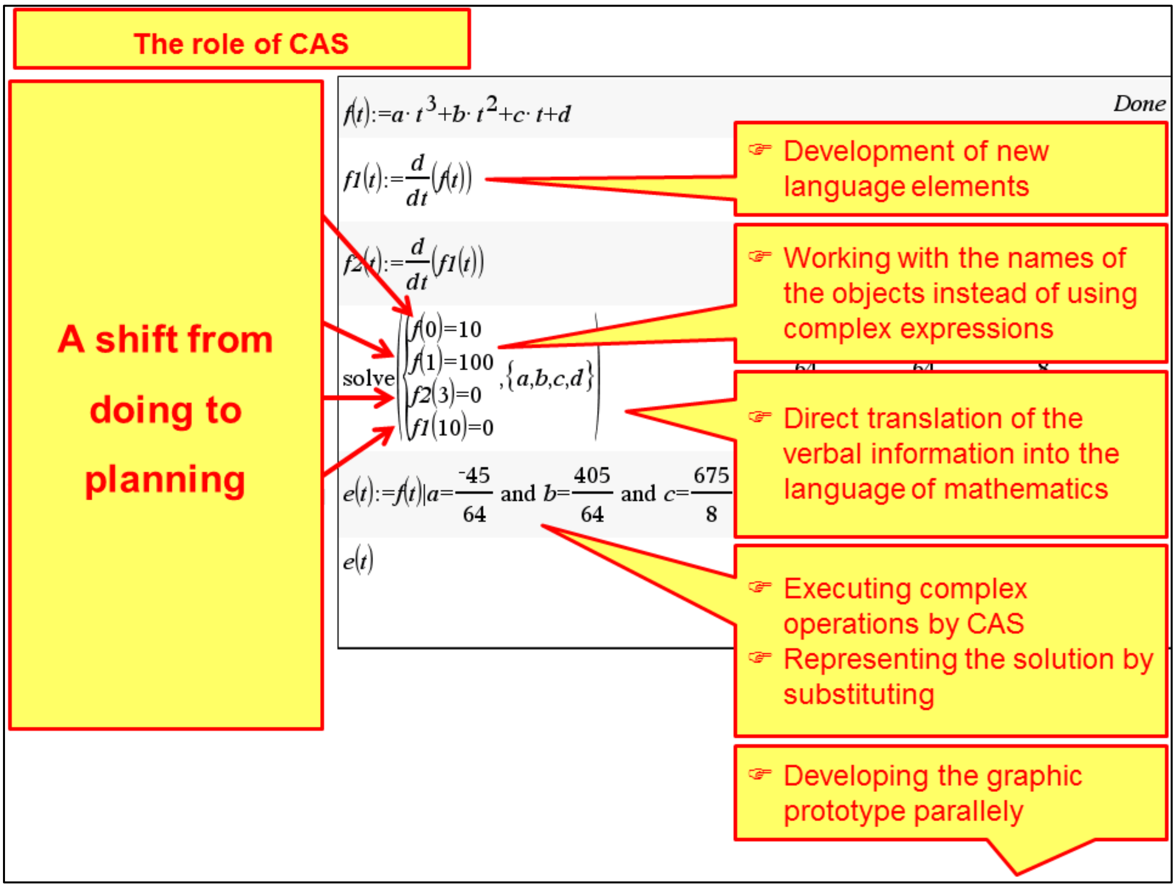
- At the beginning of the study 10 persons are down with influenza
- After one day the number has increased to 100 persons
- At the third day the growth rate is strongest
- At the 10th day the flu wave (the number of sufferers) has is maximum

Task:

a) Determine the equation of the function E and draw the graph

b) Find the day where the progressive increase of the number of sufferers turns into a degressive increase

The role of CAS:



Conclusion 3.1: The direct translation of verbal formulated decisions and arguments into symbolic representations by using CAS causes a new quality of interactions between two modes of representation. A consequence is a more stable mental model which is better suited for problem solving.

3.2 Interaction between graphic representations and CAS

A short look at results of brain research: *"2,5 million nerve fibers are connecting several parts of the body with the brain. 2 million of them are belonging to the optic nerves"* [Hanisch, Sattelberger, 2008]

A consequence is that human beings are perfectly equipped for visualization. The graphic mode of representation or more general depictive representations play an important role for the concept of multiple representations.

But the creation of a depictive representation is not sufficient for successful cognitive problem solving.

Operating on depictive representations implies a close interaction between description and depiction because procedures on a representation are usually guided by a descriptive representation. Accordingly learners should be taught to closely interconnect descriptive and depictive representations when solving problems.

Mathematical procedures and operations are executed in the symbolic mathematical language and therefore CAS are the appropriate descriptive modes of representation.

Example 3: Upper and lower sums

Source Gertrud Aumayr

At first we look at strictly monotonic increasing functions.

Given is the function f1: $f1(x) = 0.2 \cdot x^2 + 1$

The idea of this prefabricated applet:

Graphic representation: After definition of a slider for n the rectangles of the upper and the lower sums are drawn in the interval [0,6]

CAS representation in the notes application: Using the sum-operator the product sums for the upper and lower sum are calculated. For comparison also the definite integral is calculated as a black box.

Task:

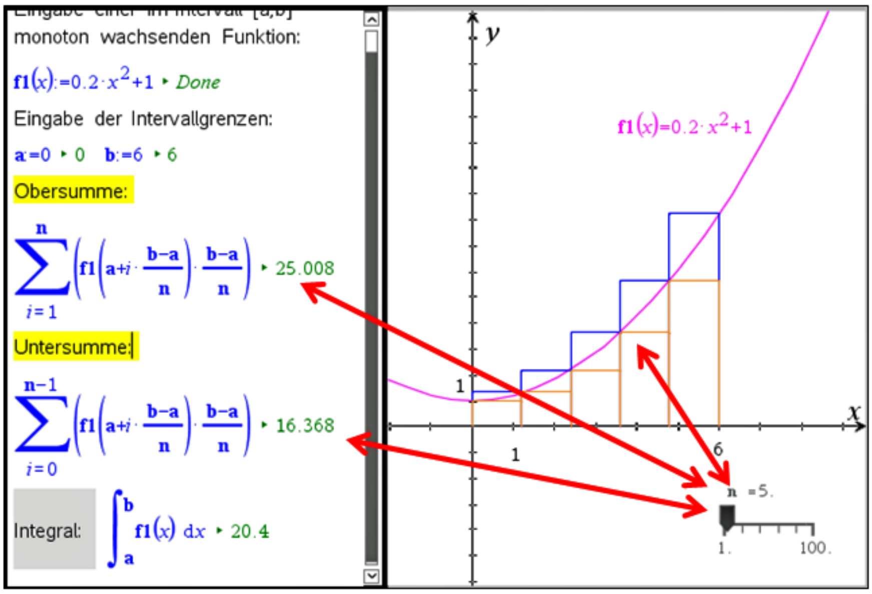
Watch the development of upper- and lower-sums for growing n look at the product sums calculated by CAS as well as the graphic representation.

Solution:

Modern digital media offers the strategy of

Dynamic linking:

Graphical manipulations as well as algebraic manipulations can simultaneously be transferred in other modes of representations. Consequences of manipulations in the algebraic window can directly be observed in the graphic window and vice versa.

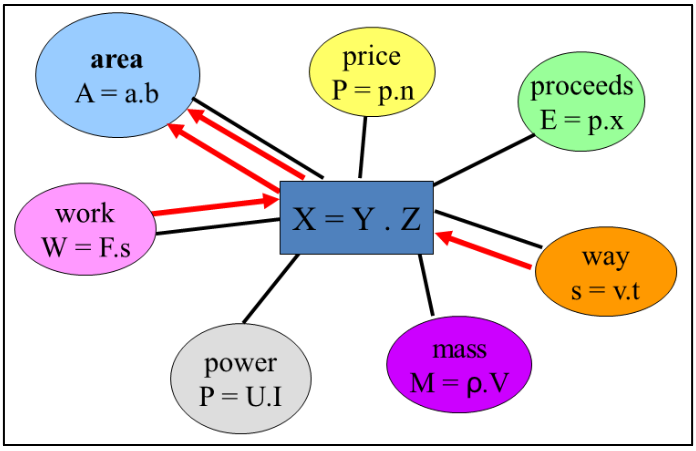


Dynamic linking:

Modifying the number n of the rectangles with the slider in the graphic window initiates the equivalent change of the product sum in the algebraic window.

Didactical comment:

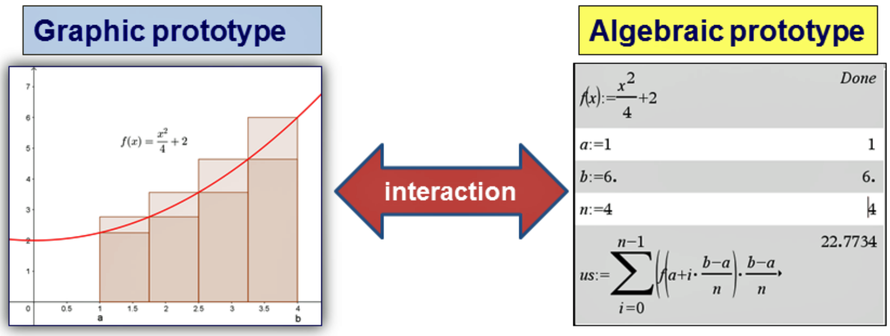
The development of the concept of the definite integral normally starts with the interpretation as an area. But after having generated first characteristics of the concept of a definite integral we should better forget this interpretation because the power of integrals are the many several interpretations in many other contexts.



The power of Integral ⇔ the relationship of many applications to the area because of a common mathematical model

The abstract interpretation of the definite integral is the limit of productsums

Depictive representation Descriptive representation



For this important cognitive step of abstraction the interaction between these modes of representation is very important

Example 4: Ship collision?

The HMS Serena and the HMS Magic are sailing on the Atlantic Ocean: Their progress is being monitored by radar tracking equipment which is geographically oriented (W-E/N-S). As they come onto the observer´s rectangular screen, the HMS Serena is at a point 900 mm from the bottom left corner of the screen along the lower edge. The HMS Magic is at a point 100 mm above the lower left corner along the left edge. One minute later the positions have changed as follows:

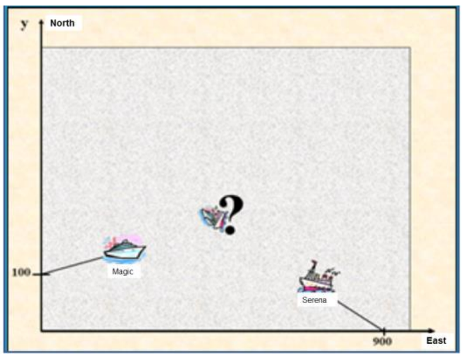
- The HMS Serena has moved to a location on the screen that is 3 mm west and 2 mm north of the previous location and
- the USS Magic has moved 4 mm east and 1 mm north.

Task:

a) Will the two ships collide if they maintain their speeds and remain on their respective course? If so, when?

b) If not, how close do they actually come to each other?

Note: 1 mm ≙ 100 m



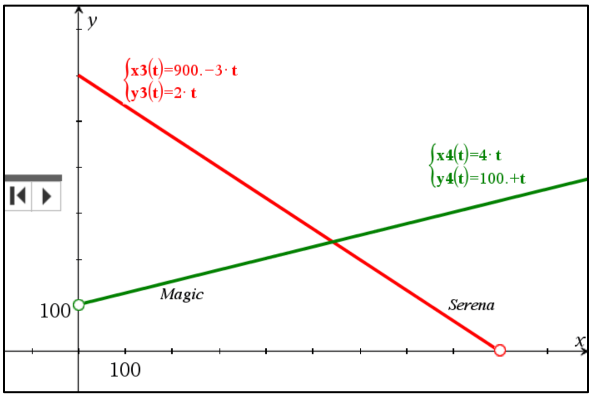
Solution:

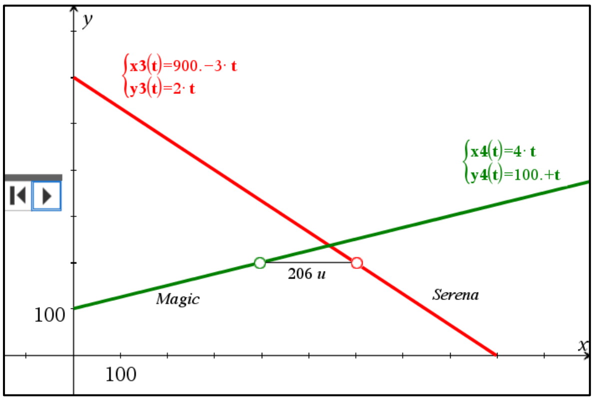
$serena(t):=\begin{bmatrix}900.\\0\end{bmatrix}+t\cdot\begin{bmatrix}-3\\2\end{bmatrix}$	*Done*
$magic(s):=\begin{bmatrix}0\\100.\end{bmatrix}+s\cdot\begin{bmatrix}4\\1\end{bmatrix}$	*Done*
$\text{solve}(serena(t)=magic(s),\{t,s\})$	$s=136.36364$ and $t=118.18182$
$serena(118.18182)$	$\begin{bmatrix}545.45454\\236.36364\end{bmatrix}$
$magic(136.36364)$	$\begin{bmatrix}545.45456\\236.36364\end{bmatrix}$

When calculating in the algebraic window the intersection point of the paths of the two ships a possible conclusion is: At this point the ships will collide.

But students have to ask: Are they at this point at the same time? And: “What is the physical interpretation of the parameters? The answer to the second question is: The parameters s and t represent the time (in minutes)

Now a shift to a depictive representation of the problem is helpful. Modern tools like TI Nspire CAS cannot only draw graphs of the paths they can also simulate the movement of the two ships with respect to the time t and show their distance.



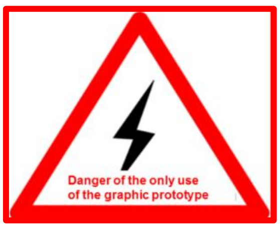


The simulation shows: The ships are not at the intersection point at the same time.

The exact calculation of the velocity of the two ships and their minimal distance needs a return to a descriptive interpretation in the algebraic window:

$\text{norm}(serena(1)-serena(0))$	3.6055513
$\text{norm}(magic(1)-magic(0))$	4.1231056
$3.6055513\cdot 6$	21.633308
$v_{Serena}=3.6055513\cdot 6$	$v_{Serena}=21.633308$
$v_{Magic}=4.1231056\cdot 6$	$v_{Magic}=24.738634$

$\text{norm}(serena(t)-magic(t))$	$5\cdot\sqrt{2\cdot(t^2-256.\cdot t+16400.)}$
$dist(t):=\text{norm}(serena(t)-magic(t))$	*Done*
$dist1(t):=\frac{d}{dt}(dist(t))$	*Done*
$\text{solve}(dist1(t)=0,t)$	$t=128.$
$dist(128)$	28.284271
$28.284271\cdot 100$	2828.4271



The only creation of a depictive representation such as drawing is not sufficient for successful cognitive problem solving even the drawing is correct. Sometimes the specific perceptual structure or other perceptual attributes can obscure the relevant structural attributes.

Example 5: What is $\sqrt{2}$?

Source: Applet "Integrator" developed by H.-J. Elschenbroich, H. Langlotz and G. Aumayr

To find an answer a definition of irrational numbers formulated by R. Fischer is helpful:

> *The step to irrational numbers takes place by declaring the possibility of an arbitrary approach as a number*

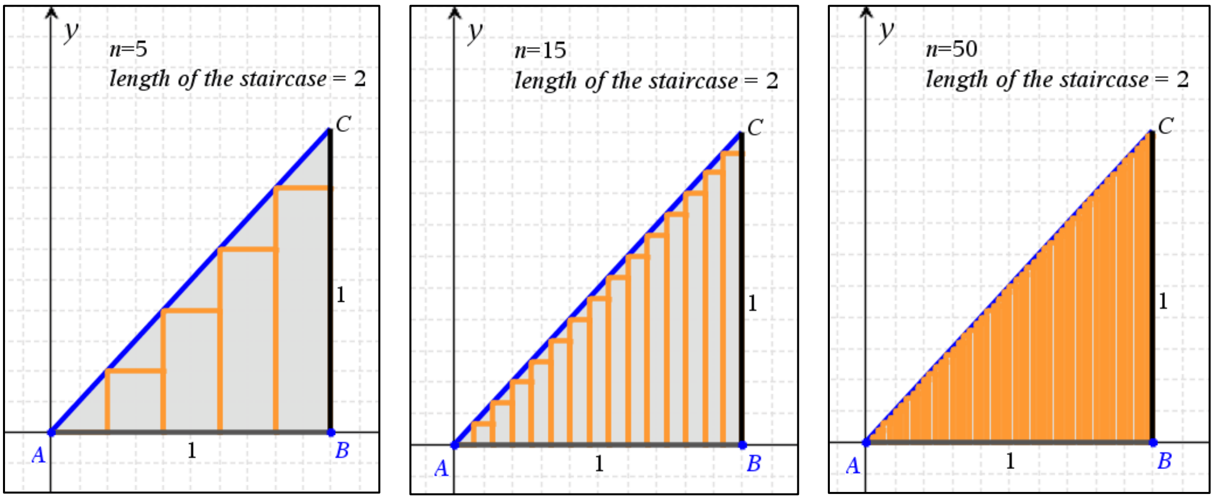
We can simulate this arbitrary approach by using the following depictive representation:

Task: What is $\sqrt{2}$?

We draw the graph of f(x)=x in the interval [0,1]. By using this applet we can draw the lower sum with respect to the number n. Now a staircase can be seen.

The central question is: "What is the length of the staircase?"

With a slider the number n can be enlarged. For growing n we again ask: "What is the length of the staircase?"



The graphic representation shows:

- For any n the length of the stair is 2
- For arbitrary large n the staircase approaches more and more to the diagonal of the square with sidelength 1.

Conclusion:

- We have a constant sequence with elements 2 ⇨ the limit is 2
- The graphic representation shows: The staircase with constant length 2 "converges" to the diagonal with length $\sqrt{2}$
- **We have shown by visualization** $\mathbf{\sqrt{2} = 2}$

Conclusion 3.2: The Interaction between graphic representations and CAS is an important contribution for building effective mathematical mental models which are the corner-stones of problem solving.

3.3 Interaction between graphic representation, CAS and data represented in interactive tables

Solving more complex problems often needs the interaction between mor than two modes of representation.

Example 6: A computer virus

Within two hours nearly all PCs in a company with 4500 networked PC-working stations are attacked by a virus. After a complete cleanup the network administrator can reconstruct the time flow:

Time (minutes)	0	20	40	60	80	100	120
Number of infected PCs	15	99	598	2346	4024	4435	4492

Task:

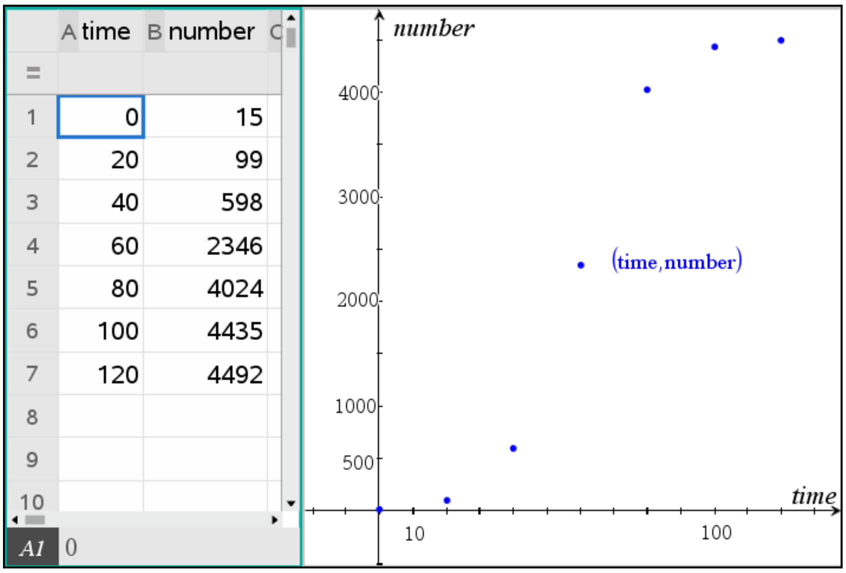
(1) What function could describe the growth process at the best?

(2) After how many minutes a maximum rate of change can be observed?

Solution:

Step 1: Entering the given data in the „Lists&Spreadsheet“ application

Step 2: Drawing the graph of the scatter plot



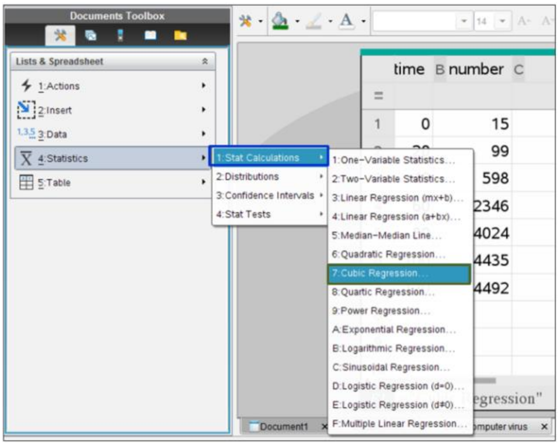
The descriptive representation (1) of the table allows no conclusion about the function type.

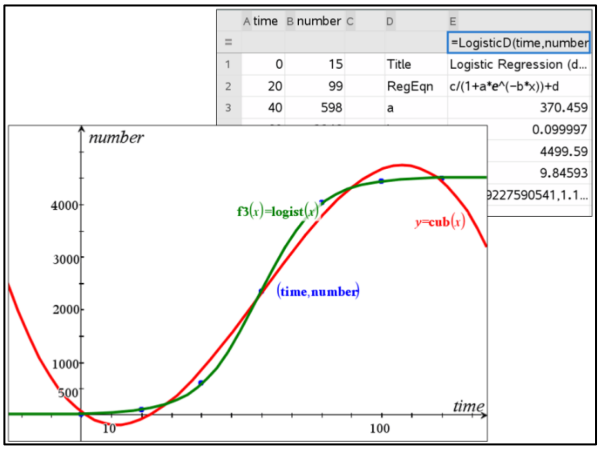
⇨ the depictive representation (2) of the graph of scatter plot is necessary

Now assumptions about the function type are conceivable, e.g.

- a polynomial function with grade 3 or
- a logistic function.

Step 3: Looking for the equations of the suitable regression functions

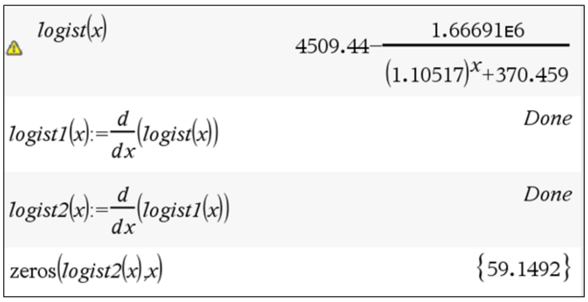




Tools like TI Nspire offer the descriptive representations (3) of the function equation together with parameters like R^2 the coefficient of determination. The depictive representation (4) of the regression functions together with the given parameters lead to the solution of the problem.

Task (2):
After how many minutes a maximum rate of change can be observed?

Now the descriptive representation of CAS is necessary.



Conclusion3.3: A solution of such a problem is only thinkable with the interaction of various descriptive and depictive representations. If algebraic operations are necessary the descriptive representation mode of CAS is important.

3.4 Interaction between recursive prototypes, graphic representations and CAS

The only new content in our mathematics curriculum caused by the use of technology is the representation of functional dependences by recursive models. Thereby a large area of growth processes becomes accessible for mathematics education which before only could be solved with complex differential equations. Also for the development of the concept of irrational numbers the use of recursive models is very helpful.

But the only use of the depictive representation of the result of the simulation process can only lead to assumptions. For ensured conclusions about characteristics of the functions like convergence a.s.o. descriptive representations which are possible by using CAS are necessary.

Example 7: A recursive model for the approximation of irrational numbers [Schweiger, 2013]

Given are two recursive models for the approximation of $\sqrt[k]{a}$

Model 1: $x_{n+1} = \frac{1}{2}\cdot\left(x_n + \frac{a}{x_n^{\,k-1}}\right)$ and Model 2: $x_{n+1} = \frac{1}{k}\cdot\left((k-1)\cdot x_n + \frac{a}{x_n^{\,k-1}}\right)$

Task: Investigate the convergence of the two models of $\sqrt[k]{a}$

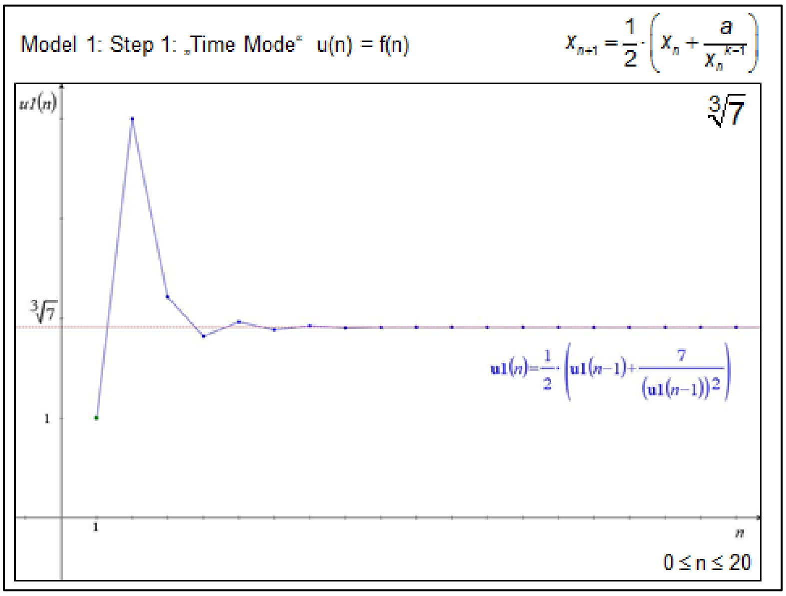
The learning process proceeds in two phases:

- **Phase 1** (experimental phase): Draw the graphs of the sequence in the “time mode” and the “web mode”. Is a convergence observable?
  In this experimental phase depictive representations of the graphs lead to conjectures.
- **Phase 2** (exactifying phase): Calculate the fixed points of the sequences and investigate the character of the fixed point by using the fixed point theorem.
  A secured answer is only possible with a proof by symbolic operations ⇨ the interaction between the graphic representation and the algebraic representation offered by CAS is necessary.

Phase 1 (experimental phase): Investigating the graphs, coming to suppositions:
Students can use two representation modes of the graph:

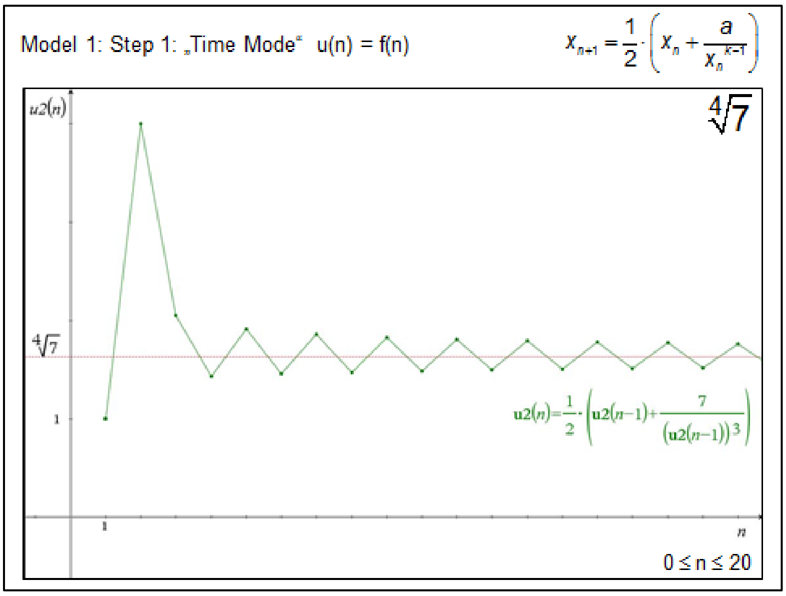
- Step 1: They use the „Time Mode“ u(n) = f(n)
- Step 2: They use the „Web Mode“ u(n) = g(u(n-1))

For the experimental phase I have chosen: a = 7 and k = 3, 4, 5



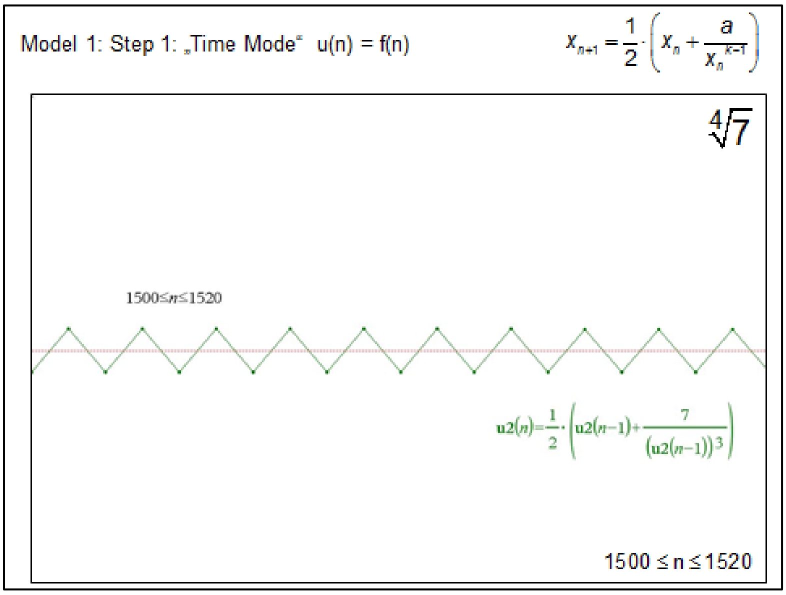
k = 3:

Model 1 seems to be convergent



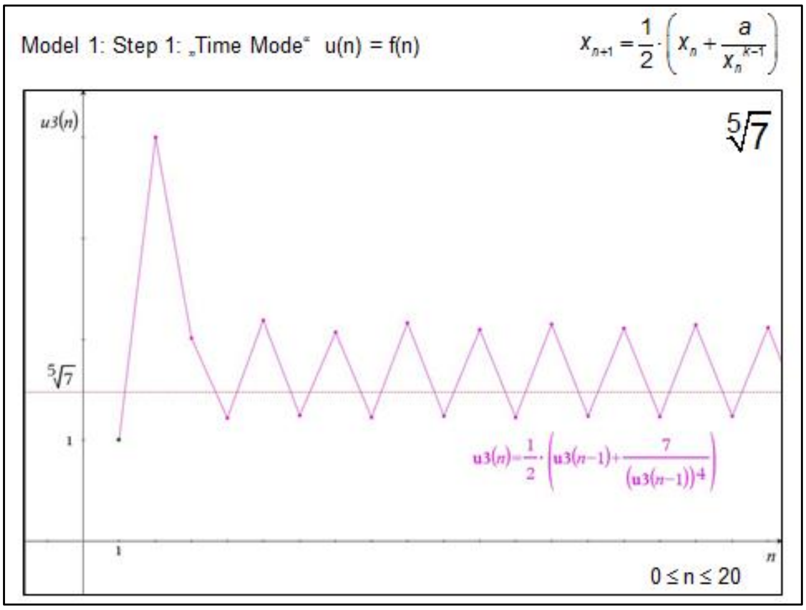
k = 4:

Observing the interval [1,20] we cannot be sure if model 1 is conver-gent. But by changing the window variables we look at other regions.



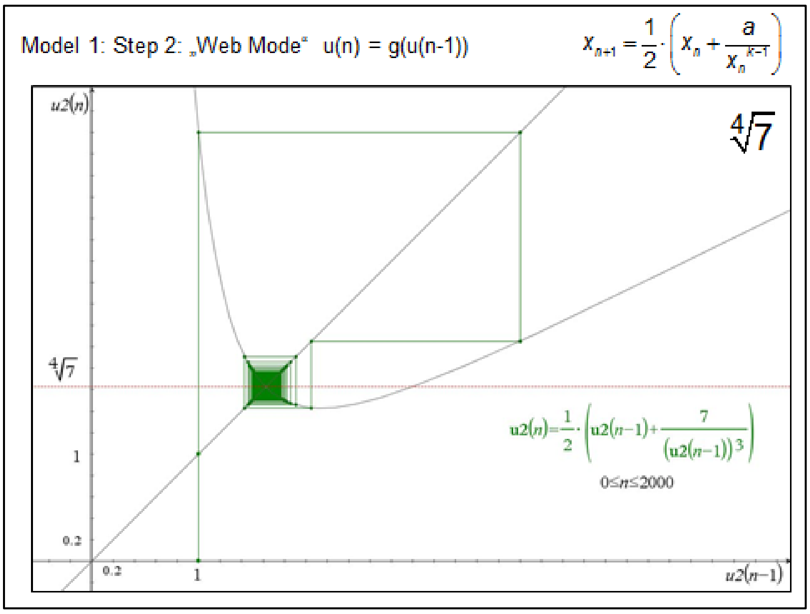
k = 4:

Let as look at the interval [1500, 1520]. We still cannot decide the convergence of model 1.



k = 5:

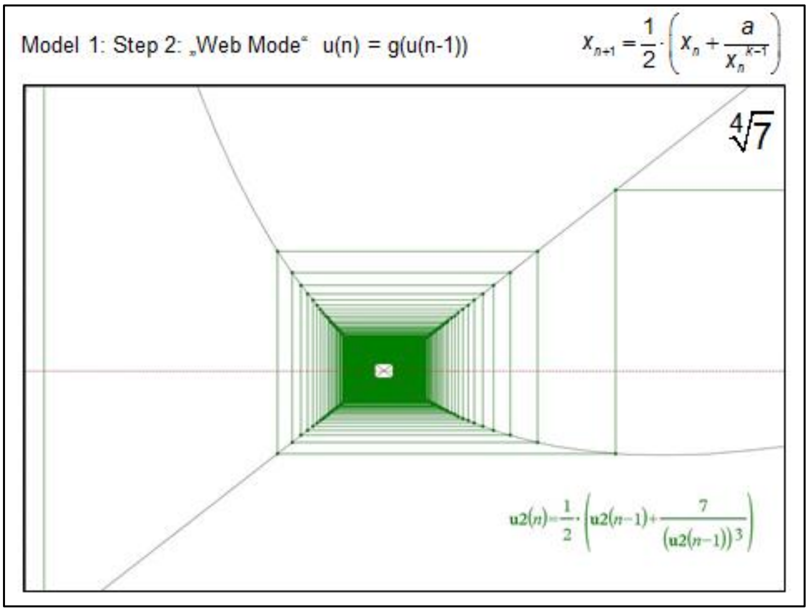
It is improbable that model 1 converges



Let us look at the **"web-mode"**:

k = 4:

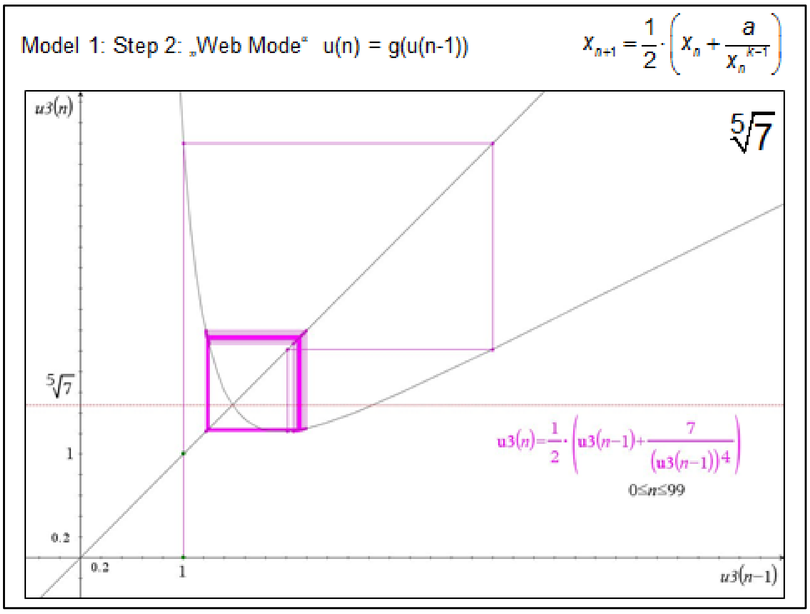
Model 1 seems to be convergent



k = 4:

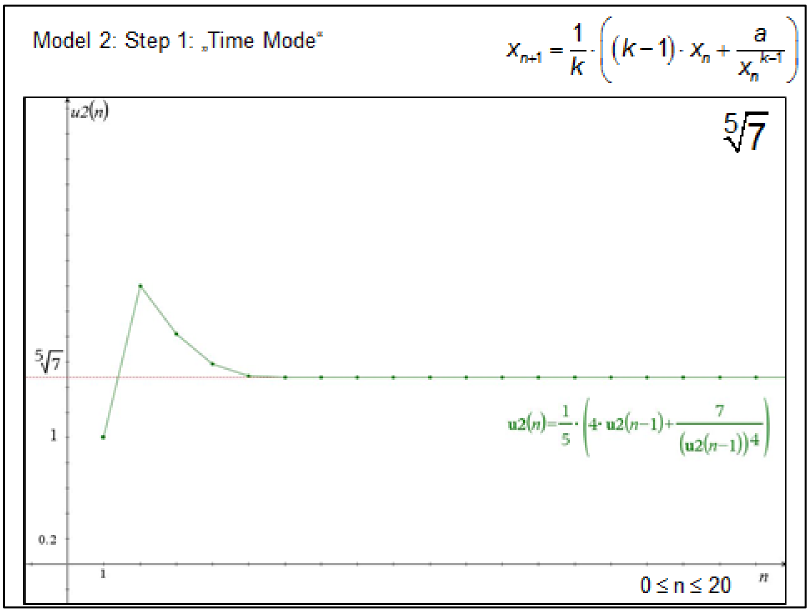
When being critical we we zoom in and now we see it will probably not be convergent.

But we have still no sureness. We have only visualized the interval [0,2000].



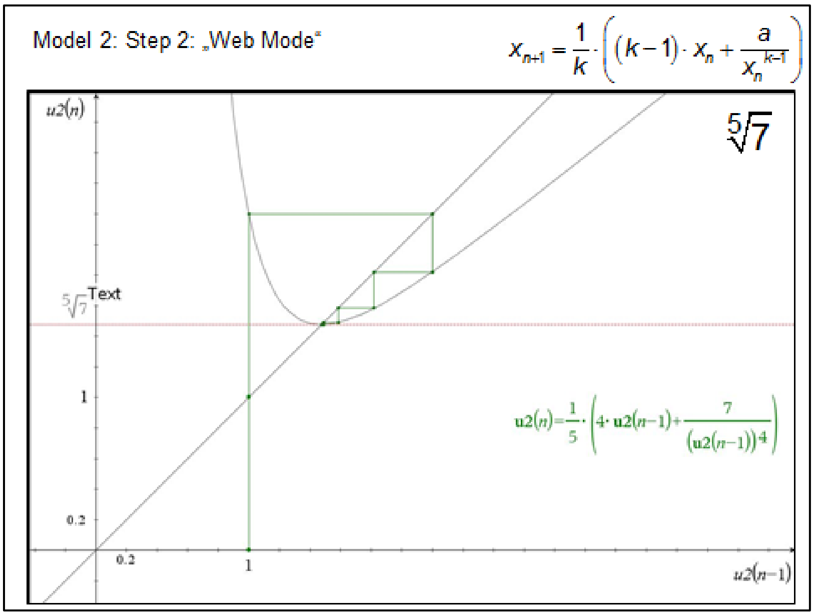
k = 5:

The graph in the web mode strengthens the supposition that model 1 is not convergent.



k = 5:

Model 2 seems to be convergent also for k=5.



k = 5:

The web graph strengthens the supposition.

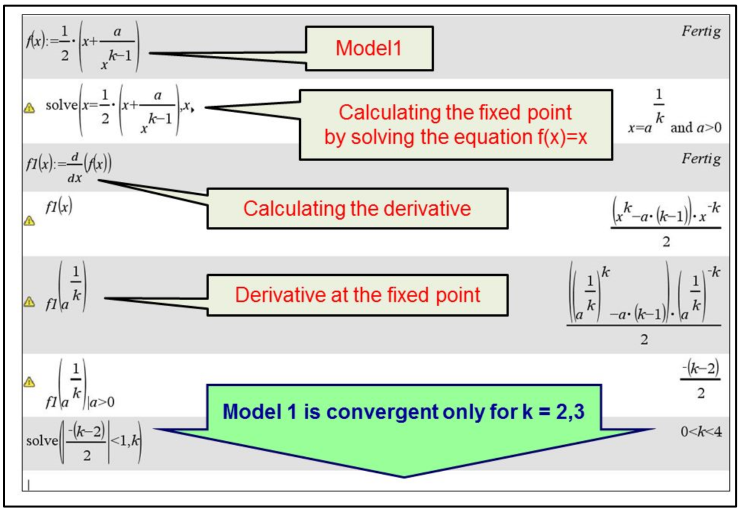
Phase 2 (exactifying phase): Proving the convergence of the given sequences

I think this example clearly shows the possibilities but also the limits of the use of technology in the experimental learning phase or we can say often an experimental phase is only possible by technology. But we cannot be content with these results, in an exactifying phase the power of CAS is necessary to get certainty.

At first we need **some theoretical prerequisites**:

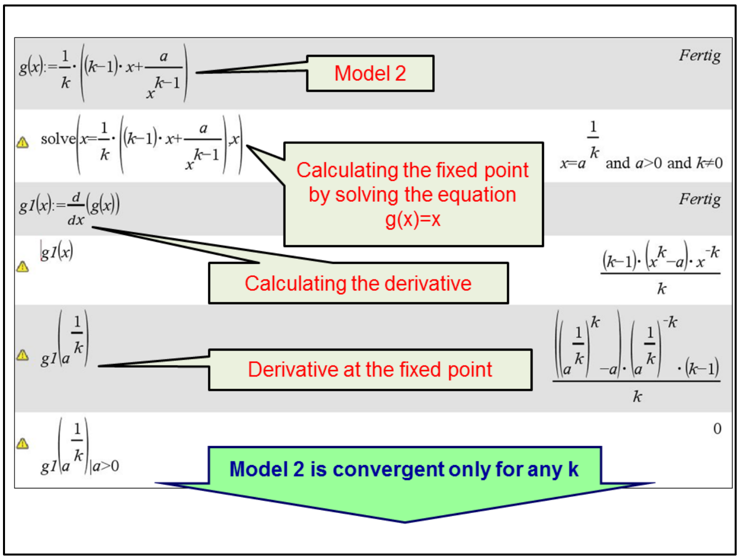
- **A fixed point x*** of a function f is an element of the function's domain that is mapped to itself, that is $f(x^*) = x^*$.
- A fixed point is **an attractive fixed point x*** of a sequence given by a difference equation $x_n = f(x_{n-1})$, if f converges to x*, that is $\lim_{n\to\infty} f(x_n) = x^*$.
- **The fixed point theorem:** A fixed point x* of a difference equation *$x_n = f(x_{n-1})$ (f is continuous and differentiable) is an attractive fixed point, if* $\left|f'(x^*)\right| < 1$ *and is distractive, if* $\left|f'(x^*)\right| > 1$ *.*

The task of phase 2 is: Calculate the fixed points and investigate the character of the fixed point by using the fixed point theorem.



Without knowing the condition that the radicand a must be positive students will not get suitable results.

The CAS tool is critical: When looking at the solution of the equation f(x)=x we will find the condition which the CAS tool expects: **a>0**.



Results:

Model 1 is convergent only for k = 2,3

Model 2 is convergent only for any k

Conclusion 3.4 By simulating technology offers a depictive representation of the recursive model which can be used for verbal descriptions of the functional dependence or for conjectures. But exact results or proves of conjectures need the close interaction with symbolic representations offered by CAS.

Summary

I tried to express the relevance of following thesis:

- The principle of multiple representations, the combined use of different representations, is a key strategy for teaching and learning mathematics.
- Computer based multimedia learning differ from traditional learning environments because several modes of representations are available simultaneously. This fact and the possibility of a direct exchange of information between the several representations ("dynamic linking") encourages the development of a stable mental model.
- CAS are the best modes of representations to express the symbolic language of mathematics because these symbolic representations allow not only the depiction of symbolic objects but also the performance of mathematical operations.
- **In a computer-based learning environment CAS are indispensable for realizing the principle of „Multiple Representations".**

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