

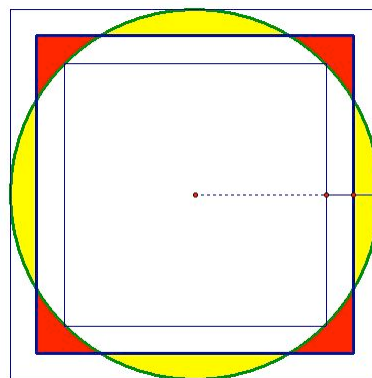
Squaring the circle



Adrian Oldknow is Emeritus Professor of Mathematics and Computing Education at the University of Chichester, and visiting fellow at the London University Institute of Education. He chairs the Mathematical Association's ICT subcommittee, is a member of its Professional Development Committee and a member of Council. He is Treasurer of the Joint Mathematical Council and a member of ACME's 'Outer Circle'.

In this article Adrian Oldknow describes how the classic geometry problem of “squaring the circle” can be approached using the TI-Nspire. In the first part of the article he shows how to find a good approximation to the squared circle, leading directly to an approximation for π . Another approximation technique for calculating the area of a circle is then explored.

A geometry problem studied by the ancient Greeks was how to construct a square of exactly the same area as a given circle. Nowadays we know that this is an impossible task. However, it is a myth that the Greek geometers only used abstract methods. Archimedes, in particular, was a very practical person, and he worked out many very good approximate methods. In order to test whether the area of one shape was greater or less than another he would make accurate drawings, cut them out from the same material and use a pair of balances to compare their weights, and hence their areas.



Nowadays we can use computer software to calculate areas for us, at least approximately. We can find a good approximation to the squared circle and the coloured picture gives the basic idea. The green circle lies snugly within the largest square shown, and the smallest square fits snugly inside the circle. So there must be a square between these two squares which has area equal to that of the circle. The regions coloured red are the bits of the square that are not inside the circle, and those coloured yellow are the bits of the circle that are not inside the square. So our aim is to adjust the middle sized square to make the red and yellow areas as equal as possible. We can then use the square to approximate π .

Below is an outline method for setting this up on the TI-Nspire handheld. The named points are for reference in this article but there is no need to enter them on the screen unless you wish to do so.

Draw a circle

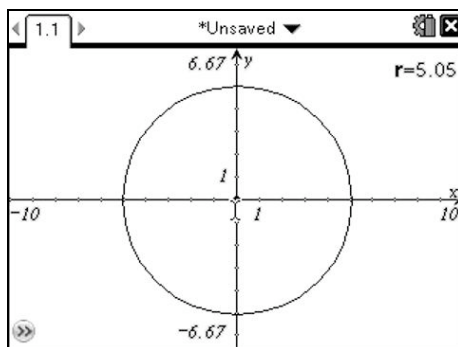
Start with a blank Graphs screen.

From the Shapes menu use the Circle tool to construct a circle centred at the origin.

From the Measurement menu choose Length and measure the radius of the circle.

Place the cursor over the measurement and store it as variable r .

Notice that you can drag any point on the circle and change the value of r .



Draw the square

Use the Angle Bisector tool on the Constructions menu to construct a line through the origin at 45° to the axes.

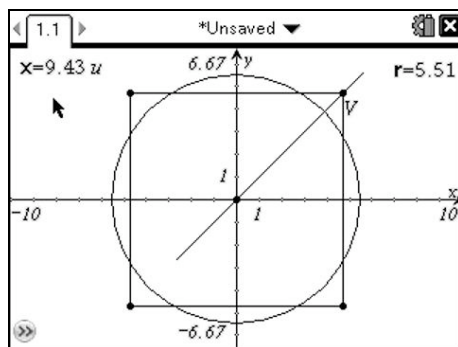
From the Points & Lines menu use the Points On tool to construct the point V.

Construct the other three corners of the square by reflecting V in the x- and y-axes.

Use the Polygon tool to construct the square through V.

Measure the length of the side of the square and store it as variable x .

Notice that you can drag point V along the line and change the value of x .



Calculate the red and yellow areas

We will restrict ourselves to getting the handheld to measure areas that we could calculate for ourselves, such as squares, triangles and trapezia. As a first approximation we can define some polygons whose areas approximate the red and yellow regions of our diagram.

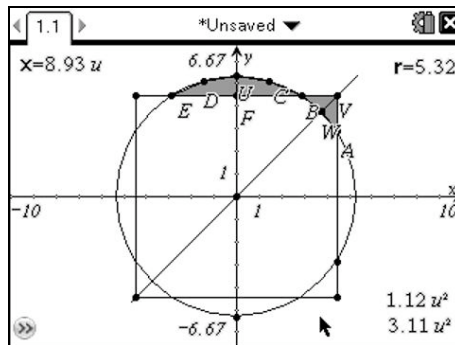
First define the quadrilateral (kite) AWBV shown below.

Use the Intersection Points(s) tool to determine the points and then the Polygon tool to define the polygon. Use the contextual menu to change its Fill Colour to grey.

We could use the triangle BEU to approximate the yellow area, but we could get more accuracy if we construct another couple of points C, D on the circle and use the polygon EDUCB instead.

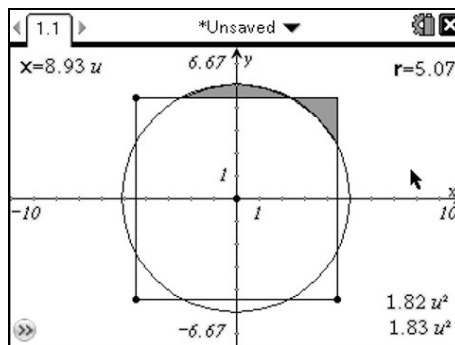
To construct point C draw the perpendicular bisector of BF and mark its intersection with the circle at C. Then D can be constructed as the reflection of C in the y-axis.

Define the Polygon EDUCB and shade it.
 Finally use the Measurement tool to display the area of the two shaded polygons.



Equalizing areas

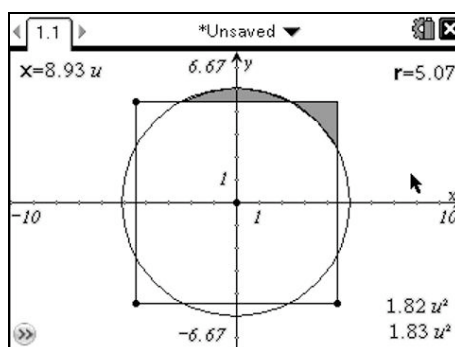
At this stage we can tidy up the figure by hiding unwanted construction lines and points. Then it is time to see if we can change the size of the circle and/or the square until the two polygons have (as accurately as possible) equal areas. It takes a little trial and improvement but here is one possible solution.



An approximation for π

We could go further with geometric explorations, but now we can just use some simple algebra and arithmetic. For a square and a circle of equal area, is there a simple relationship between the side of the square and the radius of the circle? The side of the square has length x , so its area is x^2 . Let us assume that the area of the circle is a constant p times r^2 . To find p we must solve the equation $pr^2 = x^2$ for p , i.e. $p = (x/r)^2$. We can use the Text tool to write the expression on the page, and the Calculate tool to substitute values for variables. Hence we find that the solution to squaring the circle is roughly when $p = 3.10387$.

The challenge now is to make the two shaded polygons have even more equal areas and the value of p closer to the value of π .



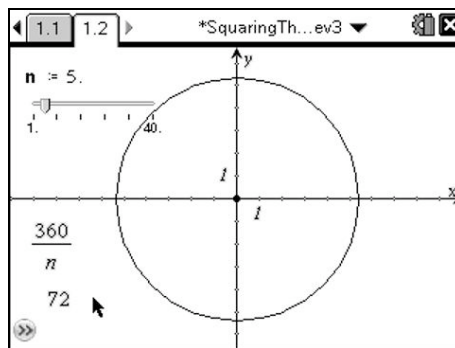
So we have used computing power, and our brains, to short-cut Archimedes' painstaking weighings. Of course we're not sure how TI-Nspire actually calculates its areas – and so in the end we are using it as a "black box". However this approach does give us a feeling for how we can use trial and improvement to solve problems approximately.

An n-sided regular polygon

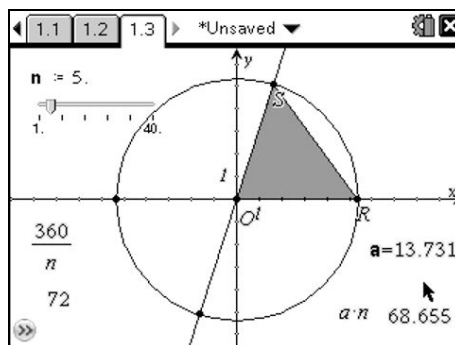
To finish let us set up a simple visual demonstration of another way to find the approximate area of a circle. Here the circle is divided into n equal triangles whose area can be calculated. As n increases the approximation gets better and better.

Here a new Graphs page has been created. We can use a slider to give us counting numbers as big as we like. From the Actions menu choose Insert Slider and assign to it the variable n . With the cursor on the slider press $\text{(ctrl)} \text{(menu)}$ and change the settings to run from 1 to 40 with a step size of 1 and an initial value of 5.

You need to check that the Document Settings (in the Tools File menu) are set for angles in Degrees. Then divide 360 by the value of n to obtain the value of an angle. From the Actions menu choose Text and enter $360/n$. Then from the Actions menu choose Calculate and evaluate the expression.

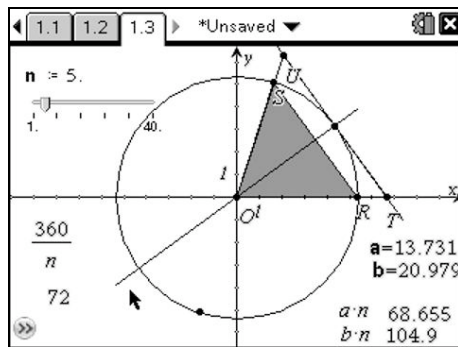


Choose Rotation from the Transformations menu and select, in turn, the x-axis, the angle 72° and the origin, O . Mark the points of intersection R and S and draw the triangle ORS . Shade the triangle, measure its area and assign it to the variable a . If we multiply this area by n we will find the area of a n -sided regular polygon that lies inside the circle. Use Text and Calculate to add this to the sheet.



Now we can use the slider to increase the value of n and investigate how the area of the n -sided polygon gets closer to the area of the circle.

The diagram below shows a further construction of a triangle OTU, part of an n-sided polygon sitting just outside the circle. The area of this polygon, $b \cdot n$, provides an upper bound for the area of the circle. As n is increased $a \cdot n$ and $b \cdot n$ get closer to each other and to the true area of the circle.



This example gives a feel for the kind of limiting processes used regularly by Archimedes, and which Newton and Leibniz formalised into the calculus techniques we know today. Why not find out some more about the geometric techniques the Ancient Greeks developed to solve problems and see if you can use TI-Nspire to model them?