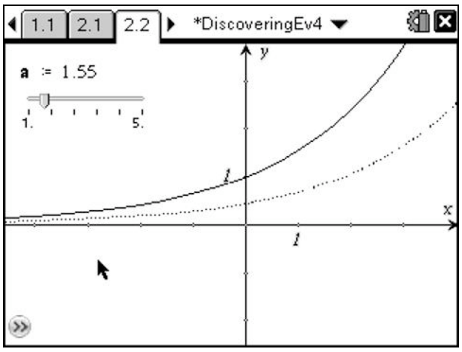
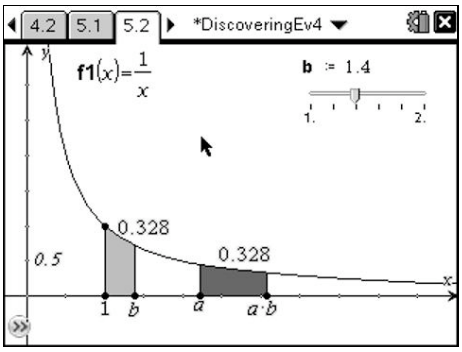
Discovering *e*

Mathematical Content:	Technical TI-Nspire Skills:
Euler's number	Manipulation of geometric constructions
Differentiation of exponential functions	Use of sliders
Integral of 1/x	Questions & answers
Sequences & series	Lists

This activity introduces Euler's number ***e*** from a variety of perspectives.

The first approach looks at how *e* can be defined as the base of the special exponential function that is its own derivative.

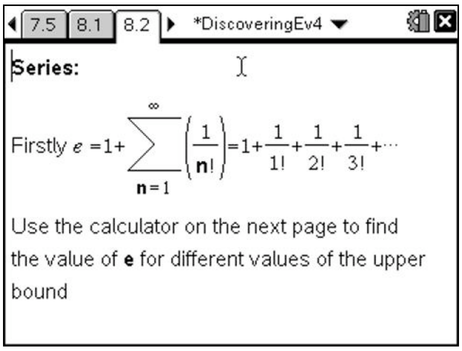


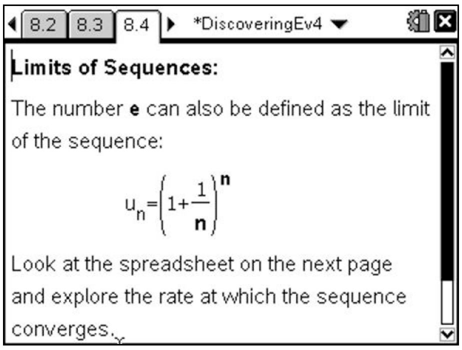


The second section explores the integration of the 1/x function. It builds through the process of observing that the area under the curve behaves like a logarithmic function, justifying this statement and then it moves towards the observation that the base of this logarithmic function turns out to be *e*.

Finally there are two extension tasks, one looking at the definition of a series whose limit is *e*, and another looking at a sequence whose limit is *e*.

As a whole, this activity should give your students insight into the mathematical origins of this interesting and pervasive number.



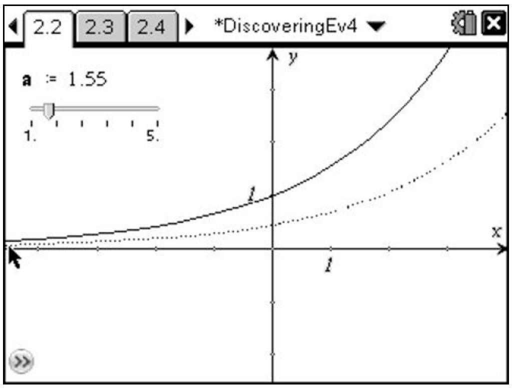


Discovering *e* – Student worksheet

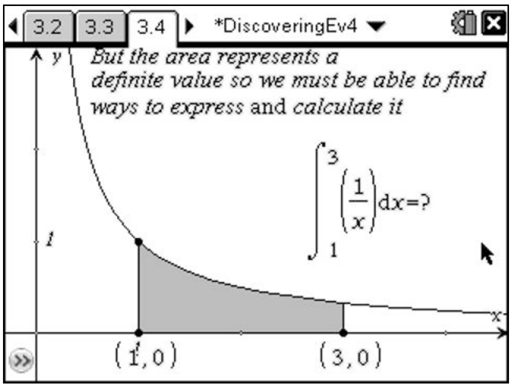
In this activity you will explore the origins of Euler's Number sometimes referred to simply as ***e***.

Task 1:

To begin the activity you will need to open the DiscoveringE.tns file on your handheld. The opening pages introduce the task and explain what to do on page 2.2. To move to the next page, press ctrl ▸.
On page 2.2 you are presented with graphs of an exponential function with base *a* and its derivative. Manipulate the slider and identify 3 different cases where the derivative is less than, greater than and equal to the function. For what value of *a* is the function equal to the derivative for all values of x?



By the end of this task on page 2.6 you will have found one definition for Euler's Number.

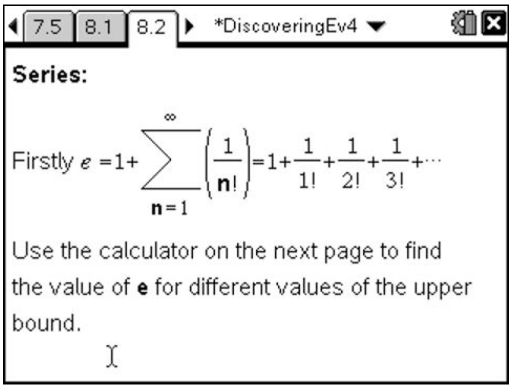


Task 2:

In this second, longer, task you will meet an alternative definition for Euler's Number this time coming from the area of integration. Starting on page 3.1, you will explore the definite integral of $\frac{1}{x}$.
Follow the instructions on pages 3.1 to 7.5 to explore the various properties of this interesting integral before finally exploring how it is related to Euler's Number.

Extension Tasks:

In this final section you will explore how Euler's Number can be defined as the limit of a series and sequence, which seem completely unrelated to the other places we have discovered Euler's Number so far.
On page 8.3 when you are looking at the Series, you can force the handheld to give you decimal (rather than fractional) answer by pressing ctrl enter, and can reuse the previous entry by pressing ▲ ▲ enter, which will copy then entry down allowing you to edit it and use a different upper limit

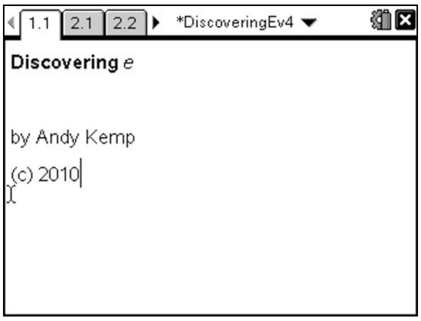


On page 8.6 if you need to move between the upper and lower panes you can do this by pressing ctrl tab.
Working through this activity should hopefully have given you some insight into the mathematical richness hidden in the seemingly innocent looking number ***e***.

Discovering e – Detailed notes

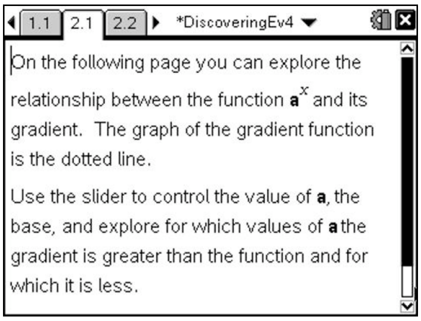
These notes briefly describe the content of each page and draw attention to any important elements

Page 1.1



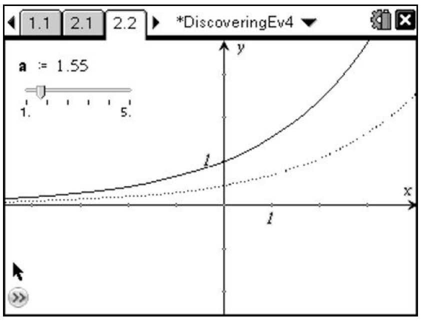
This page is simply a title page

Page 2.1



This page introduces the first task of exploring the relationship between a^x and its gradient function

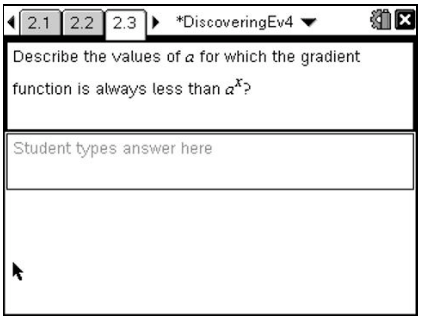
Page 2.2



On this page the student can manipulate the base of the exponential function by adjusting the slide.

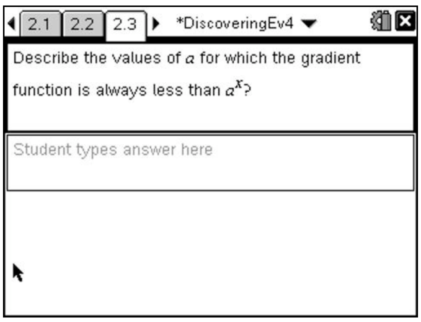
The gradient function is the dotted graph.

Page 2.3



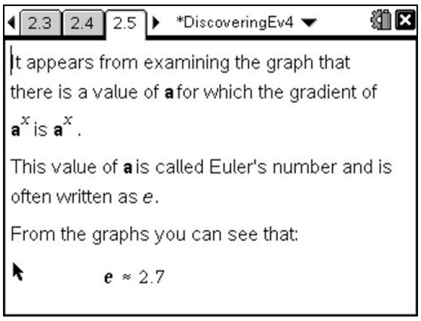
The student should observe that the gradient function is always less than the function when **a** is less than 2.7

Page 2.4



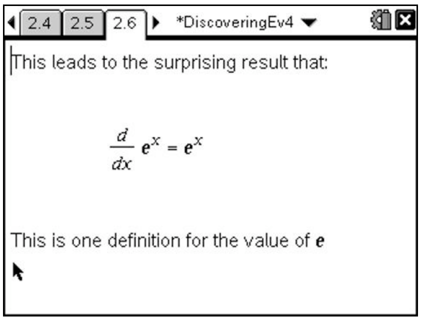
The student should observe that the gradient function is always greater than the function when **a** is greater than 2.7

Page 2.5



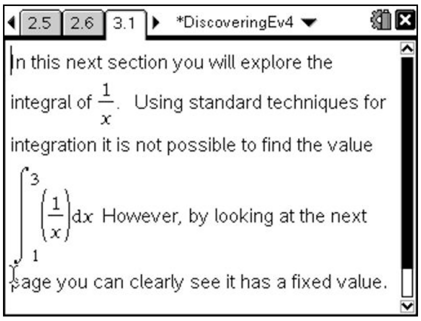
This page explains what has been discovered on the previous three pages.

Page 2.6



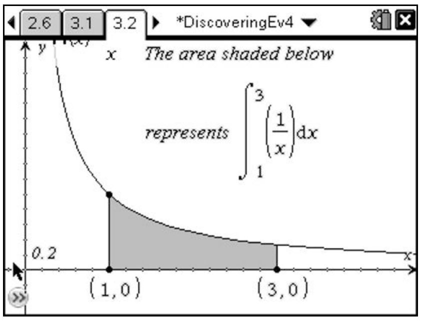
This page summarises the first definition for Euler's Number.

Page 3.1



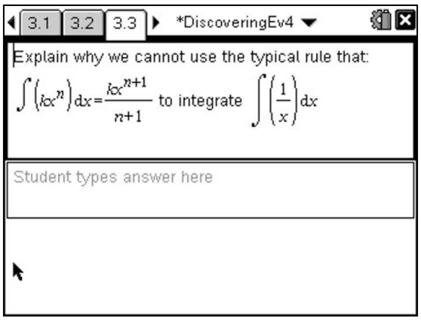
This page introduces the second task, which explores the integral of $\frac{1}{x}$.

Page 3.2



This page shows the definite integral between 1 and 3 of $\frac{1}{x}$. It is clearly a finite area.

Page 3.3

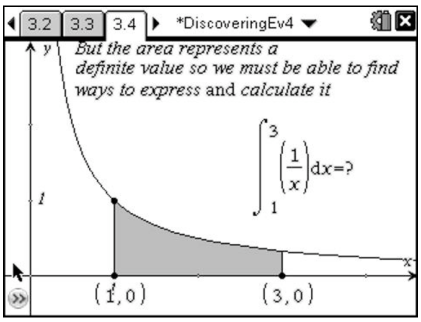


The student should hopefully observe that using the standard polynomial integral rule would lead to:

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^0}{0} = \frac{1}{0}$$

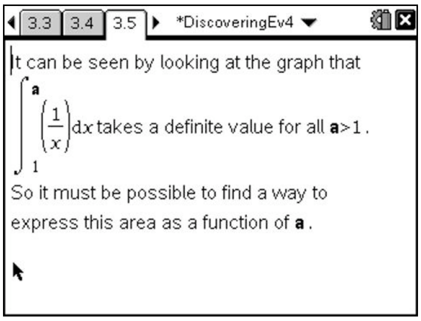
But division by zero is undefined...

Page 3.4



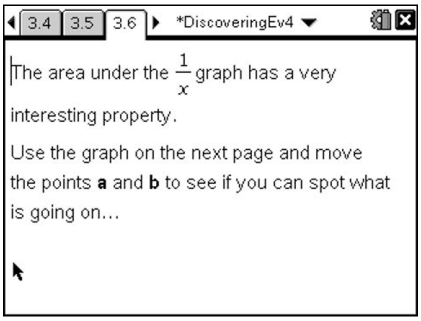
This page reminds the student that despite the standard rule leading to an undefined value the integral must have a defined value: we could measure it using a numerical method like the trapezium rule.

Page 3.5



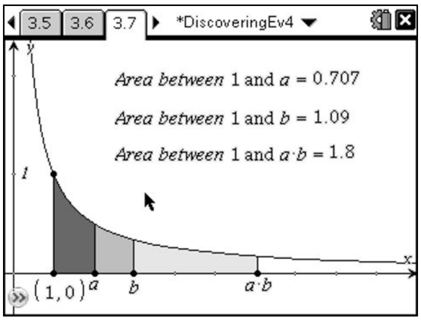
In fact the integral must have a value for any upper limit (greater than 1) as the reciprocal function is well behaved as x moves away towards infinity. So the area must remain finite for any upper finite limit.

Page 3.6



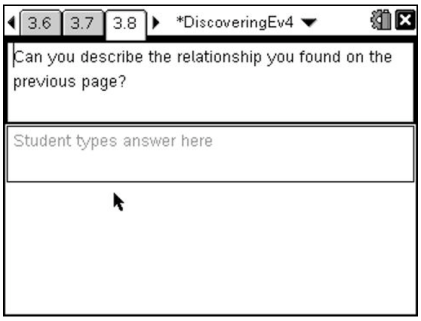
This page introduces the idea that this integral has an interesting property that may help express the area of the integral.

Page 3.7



Students should hopefully notice that, however they move the points **a** and **b**, the areas between 1 and **a** and 1 and **b** add together to give the area between 1 and **ab.**

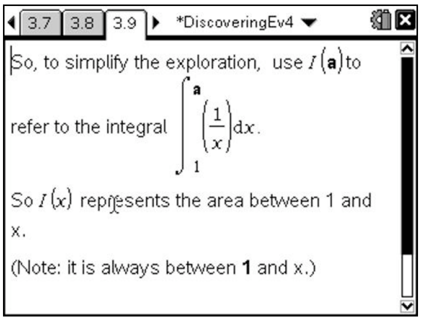
Page 3.8



Students should conclude something like:

$$\int_1^{a} \frac{1}{x} + \int_1^{b} \frac{1}{x} = \int_1^{ab} \frac{1}{x}$$

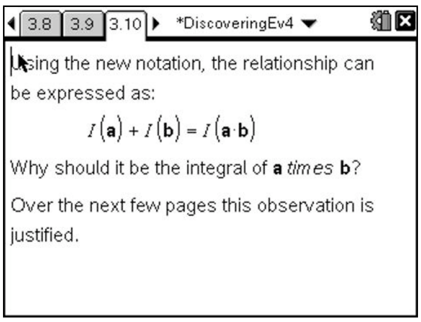
Page 3.9



At this point we simplify the notation by making:

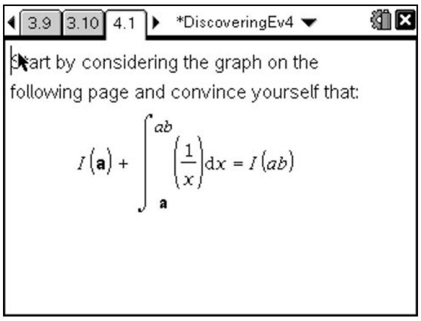
$$\int_1^{x} \frac{1}{t} dt = I(x)$$

Page 3.10



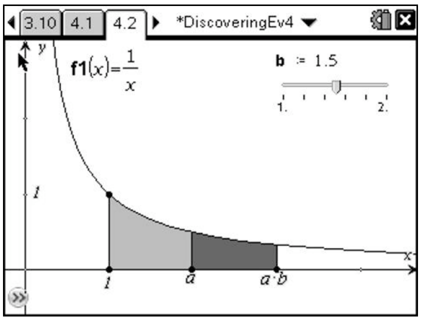
This change of notation allows us to rewrite the property expressed on page 3.8 in a slightly different form, which students may recognise as being similar to one of the properties of logarithms.

Page 4.1



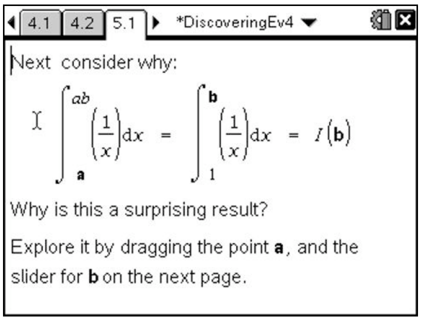
Now we begin the process of justifying the observation made on Page 3.8 and refined on page 3.10, by considering the following property,

Page 4.2



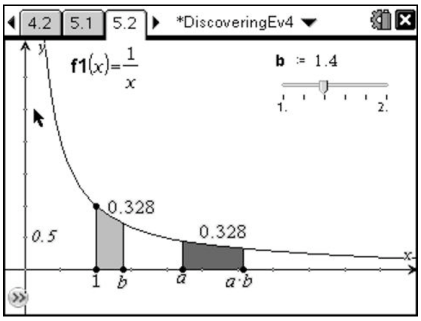
It is clear from the graph that the integral between 1 and **a** plus the integral between **a** and **ab** is equivalent to the integral between 1 and **ab**.

Page 5.1



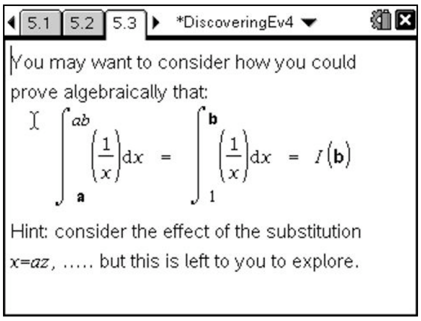
Next we will explore the following more surprising property that that the integral between **a** and **ab** is the same as the integral between 1 and **b**

Page 5.2



Graphically we can see that by dragging the point **a** about or by changing the value of **b** the two areas remain the same size.

Page 5.3

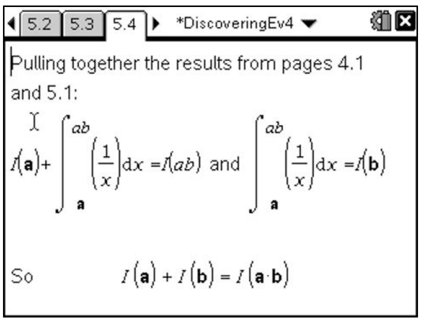


The proof of this statement is left for the student to explore independently but is included here for completeness:

If we let $x = az$ then $\frac{dx}{dz} = a$ so,

$$\int_a^{ab} \frac{1}{x}dx = \int_{x=a}^{x=ab} \frac{1}{az}\frac{dx}{dz}dz = \int_{z=1}^{b} \frac{a}{az}dz = \int_1^b \frac{1}{z}dz =$$

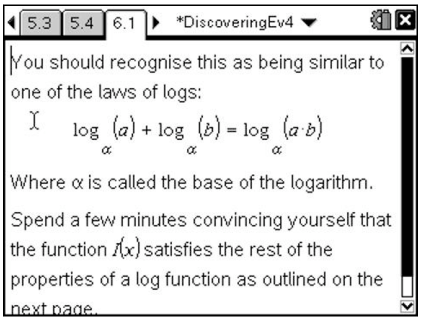
Page 5.4



This page pulls together the two properties to justify the result:

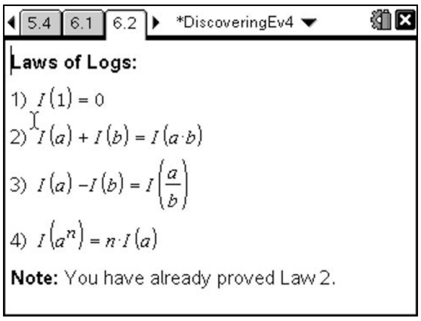
$$I(a) + I(b) = I(ab)$$

Page 6.1



This page draws the student's attention to the fact that the property they have just justified is the equivalent to one of the law of logarithms,

Page 6.2



This page outlines the other laws of logs and asks the students to justify them, proofs of these statements are included here for completeness.

1) $I(1) = \int_1^1 \frac{1}{x} dx = 0$ as the upper and lower limit are the same
2) Already completed
3) Consider $I\left(\frac{a}{b}\right) + I(b)$

   Using rule 2 on this gives:

   $I\left(\frac{a}{b}\right) + I(b) = I\left(\frac{a}{b} * b\right) = I(a)$, so

   $I(a) - I(b) = I\left(\frac{a}{b}\right)$ as required
4) This statement is easy to prove when n is an integer as:

   $I(a^n) = I(a \cdot a \cdots a \cdot a) = I(a) + \cdots + I(a) =$

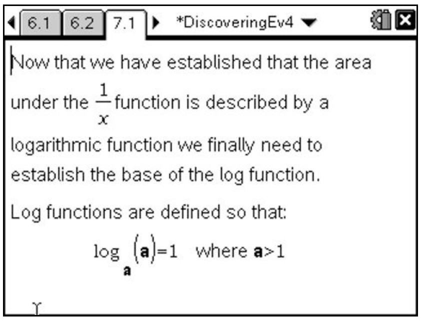
   We can show this for any rational number by letting $n = \frac{p}{q}$ where $p, q \epsilon \mathbb{Z}$ and $a^n = u$

   so:

   $q\,\mathrm{I}(u) = I(u^q) = I\left(a^{\frac{p}{q} \cdot q}\right) = I(a^p) = pI(a)$

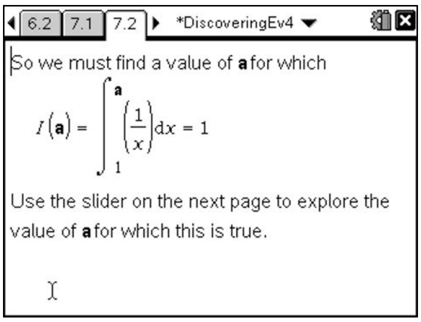
   So $I\left(a^{\frac{p}{q}}\right) = \frac{p}{q} I(a)$ as required.

Page 7.1



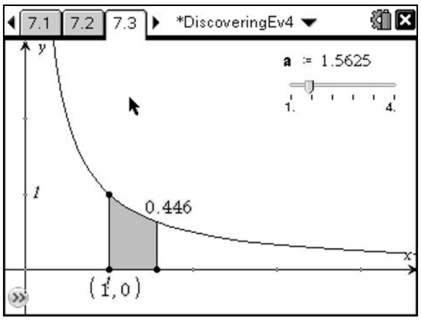
This page reminds students that the base of any logarithm is such that $\log_a a = 1$

Page 7.2



So relating this fact back to the integral leads back to the graphical environment where we must find the value of *a* such that I(*a*)=1

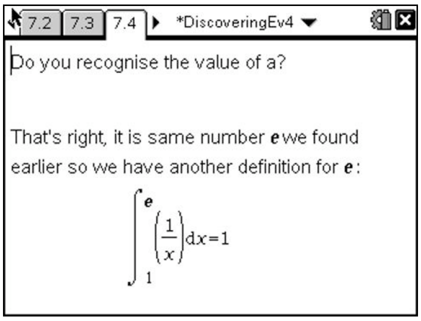
Page 7.3



On this page students should adjust the value of *a* to find a value for which the area is 1.

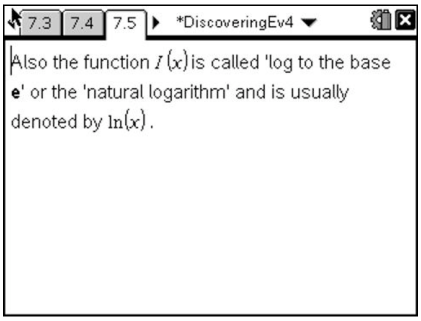
Students should find that this occurs when *a* is about 2.7

Page 7.4



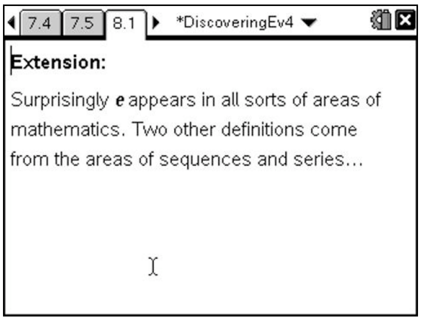
Students should hopefully remember that this value of 2.7 is the same as the value they found in the first task.

Page 7.5



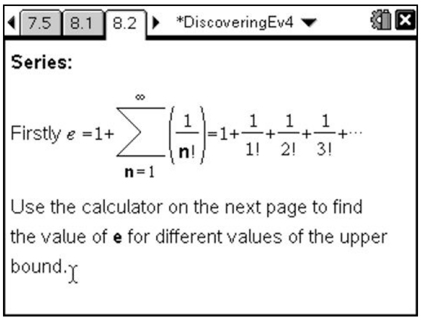
So we can conclude that the integral of 1/x (between appropriate limits) can be expressed by a log function with a base of ***e**.*

Page 8.1



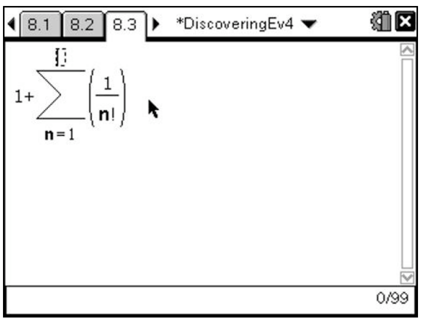
This page introduces the extension section.

Page 8.2



Another way of finding the value of ***e*** is by considering the limit of the series.

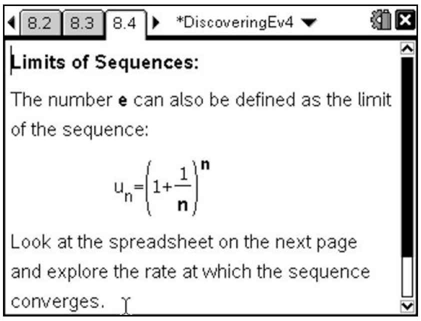
Page 8.3



On this page students are encouraged to try different upper limits to this series to see how quickly it converges to the value of ***e***

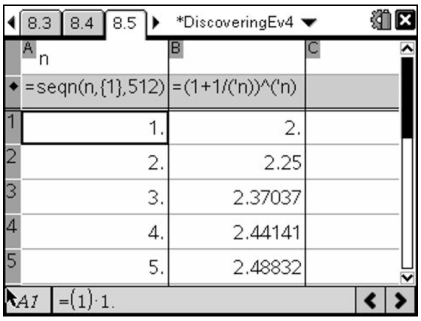
You can force the handheld to give you decimal (rather than fractional) answer by pressing ctrl enter, and can reuse the previous entry by pressing ▲ ▲ enter, which will copy the previous entry allowing you to edit it and use a different upper limit.

Page 8.4



This page suggests an alternative definition for *e* this time as the limit of a sequence as n tends to infinity

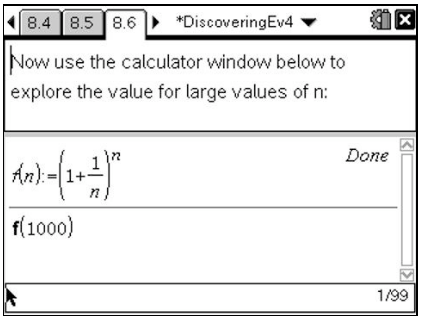
Page 8.5



On this page students can explore the rate of convergence by scrolling through the values.

Students should notice that this sequence takes a long time to get close to the value of **e** (in fact it takes over 70 terms before we even get a number starting 2.7...)

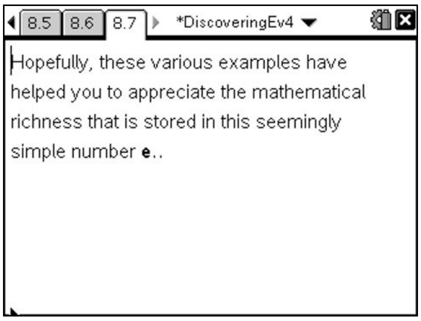
Page 8.6



On the next page we have defined a function $f(n) = \left(1 + \frac{1}{n}\right)^n$

Students can try different values of n to look at what happens to the sequence as n gets very large...

Page 8.7



This final page summarises the activity.